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THEORIA
COMBINATIONIS OBSERVATIONUM
ERRORIBUS MINIMIS OBNOXIAE

AUCTORE

CAROLO FRIDERICO GAUSS

ORDINIS GUELPHORUM ATQUE ORDINIS DANNEBROG EQUITE, BRITANNIARUM HANNOVERAEQUE REGI A CONSILIIS AULAE, OBSERVATORII REGII GOTTINGENSIS DIRECTORE, ASTRONOMIAE IN UNIVERSITATE GOTTINGENSI PROFESSORE, SOCIETATUM REGIARUM GOTTINGENSIS, LONDINENSIS, EDINBURGENSIS, HAVNIENSIS, ACADEMIARUM REGIARUM BEROLINENSIS, PARISINAE, NEAPOLITANAE, HOLMIENSIS, MONACHIENSIS, SOCIETATIS ITALICAE, CURONENSIS, ASTRONOMICAE LONDINENSIS, ACADEMIAE AMERICANAE ALIARUMQUE SOCIO.

GOTTINGAE

APUD HENRICUM DIETERICH.

1823.

THEORIA
COMBINATIONIS OBSERVATIONUM
ERRORIBUS MINIMIS OBNOXIAE.

AUCTORE

CAROLO FRIDERICO GAUSS.

PARS PRIOR.

SOCIETATI REGIAE EXHIBITA FEBR. 15. 1821.

I.

Quantacunque cura instituantur obseruationes, rerum naturalium magnitudinem spectantes, semper tamen erroribus maioribus minoribusue obnoxiae manent. Errores obseruationum plerumque non sunt simplices, sed e pluribus fontibus simul originem trahunt: horum fontium duas species probe distinguere oportet. Quaedam errorum caussae ita sunt comparatae, vt ipsarum effectus in qualibet obseruatione a circumstantiis variabilibus pendeat, inter quas et ipsam obseruationem nullus nexus essentialis concipitur: errores hinc oriundi irregulares seu fortuiti vocantur, quatenusque illae circumstantiae calculo subiici nequeunt, idem etiam de erroribus ipsis valet. Tales sunt errores ab imperfectione sensuum prouenientes, nec non a caussis extraneis irregularibus, e. g. a motu tremulo aeris visum tantillum turbante; plura quoque vitia instrumentorum vel optimorum huc trahenda sunt, e. g. asperitas partis interioris libellularum, defectus firmi-

tatis absolutae etc. Contra aliae errorum caussae in omnibus obseruationibus ad idem genus relatis natura sua effectum vel absolute constantem exserunt, vel saltem talem, cuius magnitudo secundum legem determinatam vnica a circumstantiis, quae tamquam essentialiter cum obseruatione nexae spectantur, pendet. Huiusmodi errores constantes seu regulares appellantur.

Ceterum perspicuum est, hanc distinctionem quodammodo relatiuam esse, et a sensu latiori vel arctiori, quo notio observationum ad idem genus pertinentium accipitur, pendere. E. g. vitia irregularia in diuisione instrumentorum ad angulos mensurandos errorem constantem producunt, quoties tantummodo de obseruatione anguli determinati indefinite repetenda sermo est, siquidem hic semper eaedem diuisiones vitiosae adhibentur: contra error ex illo fonte oriundus tamquam fortuitus spectari potest, quoties indefinite de angulis cuiusuis magnitudinis mensurandis agitur, siquidem tabula quantitatem erroris in singulis diuisionibus exhibens non adest.

2.

Errorum regularium consideratio proprie ab instituto nostro excluditur. Scilicet obseruatoris est, omnes caussas, quae errores constantes producere valent, sedulo inuestigare, et vel amovere, vel saltem earum rationem et magnitudinem summo studio perscrutari, vt effectus in quauis obseruatione determinata assignari, adeoque haec ab illo liberari possit, quo pacto res eodem redit, ac si error omnino non affuisset. Longe vero diuersa est ratio errorum irregularium, qui natura sua calculo subiici nequeunt. Hos itaque in obseruationibus quidem tolerare, sed eorum effectum in quantitates ex obseruationibus deriuandas per scitam harum combinationem quantum fieri potest extenuare oportet. Cui argumento grauissimo sequentes disquisitiones dicatae sunt.

3.

Errores obseruationum ad idem genus pertinentium, qui a caussa simplici determinata oriuntur, per rei naturam certis *limitibus* sunt circumscripti, quos sine dubio exacte assignare liceret, si indoles ipsius caussae *penitus* esset perspecta. Pleraeque errorum fortuitorum caussae ita sunt comparatae, vt secundum legem continuitatis omnes errores intra istos limites comprehensi pro possibilibus haberi debeant, perfectaque caussae cognitio etiam doceret, vtrum omnes hi errores aequali facilitate gaudeant an inaequali, et quanta probabilitas relatiua, in casu posteriori, cuiuis errori tribuenda sit. Eadem etiam respectu erroris totalis, e pluribus erroribus simplicibus conflati, valebunt, puta inclusus erit certis limitibus, (quorum alter aequalis erit aggregato omnium limitum superiorum partialium, alter aggregato omnium limitum inferiorum); omnes errores intra hos limites possibiles quidem erunt, sed prout quisque infinitis modis diuersis ex erroribus partialibus componi potest, qui ipsi magis minusue probabiles sunt, alii maiorem, alii minorem facilitatem tribuere debebimus, eruique poterit lex probabilitatis relatiuae, si leges errorum simplicium cognitae supponuntur, saluis difficultatibus analyticis in colligendis omnibus combinationibus.

Exstant vtique quaedam errorum caussae, quae errores non secundum legem continuitatis progredientes, sed discretos tantum, producere possunt, quales sunt errores diuisionis instrumentorum, (siquidem illos erroribus fortuitis annumerare placet): diuisionum enim multitudo in quouis instrumento determinato est finita. Manifesto autem, hoc non obstante, si modo non omnes errorum caussae errores discretos producant, complexus omnium errorum totalium possibilium constituet seriem secundum legem continuitatis progredientem, siue plures eiusmodi series interruptas, si forte, omnibus erroribus discretis possibilibus secundum magnitudinem ordinatis, vna alteraue differentia inter binos terminos

proximos maior euadat, quam differentia inter limites errorum totalium, quatenus e solis erroribus continuis demanant. Sed in praxi casus posterior vix vmquam locum habebit, nisi diuisio vitiis crassioribus laboret.

4.

Designando facilitatem relatiuam erroris totalis x, in determinato obseruationum genere, per characteristicam φx, hoc, propter errorum continuitatem, ita intelligendum erit, probabilitatem erroris inter limites infinite proximos x et $x + dx$ esse $= \varphi x . dx$. Vix, ac ne vix quidem, vmquam in praxi possibile erit, hanc functionem a priori assignare; nihilominus plura generalia eam spectantia stabiliri possunt, quae deinceps proferemus. Obuium est, functionem φx eatenus ad functiones discontinuas referendam esse, quod pro omnibus valoribus ipsius x, extra limites errorum possibilium iacentibus esse debet $= 0$; intra hos limites vero vbique valorem positiuum nanciscetur (omittendo casum, de quo in fine art. praec. locuti sumus). In plerisque casibus errores positiuos et negatiuos eiusdem magnitudinis aeque faciles supponere licebit, quo pacto erit $\varphi(-x) = \varphi x$. Porro quum errores leuiores facilius committantur quam graviores, plerumque valor ipsius φx erit maximus pro $x = 0$, continuoque decrescet, dum x augetur.

Generaliter autem valor integralis $\int \varphi x . dx$, ab $x = a$ vsque ad $x = b$ extensi exprimet probabilitatem, quod error aliquis nondum cognitus iaceat inter limites a et b. Valor itaque istius integralis a limite inferiori omnium errorum possibilium vsque ad limitem superiorem extensi semper erit $= 1$. Et quum φx pro omnibus valoribus ipsius x extra hos limites iacentibus semper sit $= 0$, manifesto etiam

valor integralis $\int \varphi x . dx$ *ab* $x = -\infty$ *vsque ad* $x = +\infty$ *extensi semper sit* $= 1$.

5.

Consideremus porro integrale $\int x\varphi x . dx$ inter eosdem limites, cuius valorem statuemus $= k$. Si omnes errorum caussae simplices ita sunt comparatae, vt nulla adsit ratio, cur errorum aequalium sed signis oppositis affectorum alter facilius producatur quam alter, hoc etiam respectu erroris totalis valebit, sive erit $\varphi(-x) = \varphi x$, et proin necessario $k = 0$. Hinc colligimus, quoties k non euanescat sed e. g. sit quantitas positiua, necessario adesse debere vnam alteramue errorum caussam, quae vel errores positiuos tantum producere possit, vel certe positiuos facilius quam negatiuos. Haecce quantitas k, quae reuera est medium omnium errorum possibilium, seu valor medius ipsius x, commode dici potest erroris pars constans. Ceterum facile probari potest, partem constantem erroris totalis aequalem esse aggregato partium constantium, quas continent errores e singulis caussis simplicibus prodeuntes. Quodsi quantitas k nota supponitur, a quauis obseruatione secetur, errorque obseruationis ita correctae designatur per x', ipsiusque probabilitas per $\varphi' x'$, erit $x' = x - k$, $\varphi' x' = \varphi x$ ac proin $\int x' \varphi' x' . dx' = \int x \varphi x . dx - \int k \varphi x . dx = k - k = 0$, i. e. errores obseruationum correctarum partem constantem non habebunt, quod et per se clarum est.

6.

Perinde vt integrale $\int x \varphi x . dx$, seu valor medius ipsius x, erroris constantis vel absentiam vel praesentiam et magnitudinem docet, integrale

$$\int xx \varphi x . dx$$

ab $x = -\infty$ vsque ad $x = +\infty$ extensum (seu valor medius quadrati xx) aptissimum videtur ad incertitudinem obseruationum in genere definiendam et dimetiendam, ita vt e duobus obseruationum systematibus, quae quoad errorum facilitatem inter se differunt, eae praecisione praestare censeantur, in quibus fac-

grale $\int xx\varphi x \,.\, dx$ valorem minorem obtinet. Quodſi quis hanc rationem pro arbitrio, nulla cogente neceſſitate, electam eſſe obiiciat, lubenter aſſentiemur. Quippe quaeſtio haec per rei naturam aliquid vagi implicat, quod limitibus circumſcribi niſi per principium aliquatenus arbitrarium nequit. Determinatio alicuius quantitatis per obſeruationem errori maiori minoriue obnoxiam, haud inepte comparatur ludo, in quo ſolae iacturae, lucra nulla, dum quilibet error metuendus iacturae affinis eſt. Talis ludi dispendium aeſtimatur e iactura probabili, puta ex aggregato productorum ſingularum iacturarum poſſibilium in probabilitates reſpectiuas. Quantae vero iacturae quemlibet obſeruationis errorem aequiparare conueniat, neutiquam per ſe clarum eſt; quin potius haec determinatio aliqua ex parte ab arbitrio noſtro pendet. Iacturam ipſi errori aequalem ſtatuere manifeſto non licet; ſi enim errores poſitiui pro iacturis acciperentur, negatiui lucra repraeſentare deberent. Magnitudo iacturae potius per talem erroris functionem exprimi debet, quae natura ſua ſemper fit poſitiua. Qualium functionum quum varietas ſit infinita, ſimpliciſſima, quae hac proprietate gaudet, prae ceteris eligenda videtur, quae abeque lite eſt quadratum: hoc pacto principium ſupra prolatum prodit.

Ill. Laplace ſimili quidem modo rem conſiderauit, ſed errorem ipſum ſemper poſitiue acceptum tamquam iacturae menſuram adoptauit. At ni fallimur haecce ratio ſaltem non minus arbitraria eſt quam noſtra: vtrum enim error duplex aeque tolerabilis putetur quam ſimplex bis repetitus, an aegrius, et proin vtrum magis conueniat, errori duplici momentum duplex tantum, an maius, tribuere, quaeſtio eſt neque per ſe clara, neque demonſtrationibus mathematicis decidenda, ſed libero tantum arbitrio remittenda. Praeterea negari non poteſt, iſta ratione continuitatem laedi: et propter hanc ipſam cauſſam modus ille

tractationi analyticae magis refragatur, dum ea, ad quae principium nostrum perducit, mira tum simplicitate tum generalitate commendantur.

7.

Statuendo valorem integralis $\int xx\varphi x . dx$ ab $x = -\infty$ vsque ad $x = +\infty$ extensi $= mm$, quantitatem m vocabimus *errorem medium metuendum*, sine simpliciter *errorem medium* observationum, quarum errores indefiniti x habent probabilitatem relatiuam φx: Denominationem illam non ad obseruationes immediatas limitabimus, sed etiam ad determinationes qualescunque ex obseruationibus deriuatas extendemus. Probe autem cauendum est, ne error medius confundatur cum medio arithmetico omnium errorum, de quo in art. 5. locuti sumus.

Vbi plura obseruationum genera, seu plures determinationes ex obseruationibus petitae, quibus haud eadem praecisio concedenda est, comparantur, *pondus* earum relatiuum nobis erit quantitas ipsi mm reciproce proportionalis, dum *praecisio* simpliciter ipsi m reciproce proportionalis habetur. Quo igitur pondus per numerum exprimi possit, pondus certi obseruationum generis pro vnitate acceptum esse debet.

8.

Si obseruationum errores partem constantem implicant, hanc auferendo error medius minuitur, pondus et praecisio augentur. Retinendo signa art. 5, designandoque per m' errorem medium obseruationum correctarum, erit

$$m'm' = \int x'x'\varphi'x' . dx' = \int (x-k)^2 \varphi x . dx = \int xx\varphi x . dx$$
$$- 2k\int x\varphi x . dx + kk\int \varphi x . dx = mm - 2kk + kk = mm - kk.$$

Si autem loco partis constantis veri k quantitas alia l ab obseruationibus ablata esset, quadratum erroris medii noui euaderet $= mm - 2kl + ll = m'm' + (l-k)^2$.

9.

Denotante λ coëfficientem determinatum, atque μ valorem integralis $\int \varphi x . dx$ ab $x = -\lambda m$ vsque ad $x = +\lambda m$, erit μ probabilitas, quod error alicuius obseruationis sit minor quam λm (sine respectu signi), nec non $1-\mu$ probabilitas erroris maioris quam λm. Si itaque valor $\mu = \frac{1}{2}$ respondet valori $\lambda m = \rho$, error aeque facile infra ρ quam supra ρ cadere potest, quocirca ρ commode dici potest error *probabilis*. Relatio quantitatum λ, μ manifesto pendet ab indole functionis φx, quae plerumque incognita est. Operae itaque pretium erit, istam relationem pro quibusdam casibus specialibus propius considerare.

I. Si limites omnium errorum possibilium sunt $-a$ et $+a$, omnesque errores intra hos limites aeque probabiles, erit φx inter limites $x = -a$ et $x = +a$ constans, et proin $= \frac{1}{2a}$. Hinc $m = a\sqrt{\frac{1}{3}}$, nec non $\mu = \lambda\sqrt{\frac{1}{3}}$, quamdiu λ non maior quam $\sqrt{3}$; denique $\rho = m\sqrt{\frac{3}{4}} = 0{,}8660254\, m$, probabilitasque, quod error prodeat errore medio non maior, erit $= \sqrt{\frac{1}{3}} = 0{,}5773503$.

II. Si vt antea $-a$ et $+a$ sunt errorum possibilium limites, errorumque ipsorum probabilitas inde ab errore o vtrimque in progressione arithmetica decrescere supponitur, erit

$$\varphi x = \frac{a-x}{aa}, \text{ pro valoribus ipsius } x \text{ inter o et } +a$$

$$\varphi x = \frac{a+x}{aa}, \text{ pro valoribus ipsius } x \text{ inter o et } -a.$$

Hinc deducitur $m = a\sqrt{\frac{1}{6}}$; $\mu = \lambda\sqrt{\frac{2}{3}} - \frac{1}{6}\lambda\lambda$, quamdiu λ est inter o et $\sqrt{6}$, denique $\lambda = \sqrt{6} - \sqrt{(6-6\mu)}$, quamdiu μ inter o et 1, et proin

$$\rho = m(\sqrt{6} - \sqrt{3}) = 0{,}7174389\, m$$

Probabilitas erroris medium non superantis erit in hoc casu $= \sqrt{\frac{2}{3}} - \frac{1}{6} = 0{,}6498099$.

III.

III. Si functionem φx proportionalem ſtatuimus huic $e^{-\frac{xx}{hh}}$ (quod quidem in rerum natura proxime tantum verum eſſe poteſt), eſſe debebit

$$\varphi x = \frac{e^{-\frac{xx}{hh}}}{h\sqrt{\pi}}$$

denotante π ſemiperipheriam circuli pro radio 1, vnde porro deducimus

$$m = h\sqrt{\tfrac{1}{2}}$$

(V. *Disquiſ. generales circa ſeriem infinitam* etc. art. 28.). Porro ſi valor integralis

$$\frac{2}{\sqrt{\pi}}\int e^{-zz}\,dz$$

a $z = 0$ inchoati denotatur per Θz, erit

$$\mu = \Theta(\lambda\sqrt{\tfrac{1}{2}})$$

Tabula ſequens exhibet aliquot valores huius quantitatis:

λ	μ
0,6744897	0,5
0,8416213	0,6
1,0000000	0,6826895
1,0364334	0,7
1,2815517	0,8
1,6448537	0,9
2,5758293	0,99
3,2918301	0,999
3,8905940	0,9999
∞	1

10.

Quanquam relatio inter λ et μ ab indole functionis φx pendet, tamen quaedam generalia ſtabilire licet. Scilicet qualiscunque ſit haec functio, ſi modo ita eſt comparata, vt ipſius valor, creſcente valore abſoluto ipſius x, ſemper decreſcat, vel ſaltem non creſcat, certo erit

λ minor vel ſaltem non maior quam $\mu\sqrt{3}$, quoties μ eſt minor quam $\frac{2}{3}$;

λ non maior quam $\frac{2}{3\sqrt{(1-\mu)}}$, quoties μ eſt maior quam $\frac{2}{3}$

Pro $\mu=\frac{2}{3}$ vterque limes coincidit, puta λ nequit eſſe maior quam $\sqrt{\frac{4}{3}}$.

Vt hoc inſigne theorema demonſtremus, denotemus per y valorem integralis $\int\varphi z.dz$ a $z=-x$ vsque ad $z=+x$ extenſi, quo pacto y erit probabilitas, quod error aliquis contentus ſit intra limites $-x$ et $+x$. Porro ſtatuamus

$$x=\psi y \text{ et } d\psi y=\psi' y.dy,\ d\psi' y=\psi'' y.dy$$

Erit itaque $\psi 0=0$, nec non

$$\psi' y=\frac{1}{\varphi x+\varphi(-x)},$$

quare per hyp. $\psi' y$ ab $y=0$ vsque ad $y=1$ ſemper creſcet, ſaltem nullibi decreſcet, ſiue, quod idem eſt, valor ipſius $\psi'' y$ ſemper erit poſitiuus, vel ſaltem non negatiuus. Porro habemus $d.y\psi' y=\psi' y.dy+y\psi'' y.dy$, adeoque

$$y\psi' y-\psi y=\int y\psi'' y.dy$$

integratione ab $y=0$ inchoata. Valor expreſſionis $y\psi' y-\psi y$ itaque ſemper erit quantitas poſitiua, ſaltem non negatiua, adeoque

$$1-\frac{\psi y}{y\psi' y}$$

quantitas poſitiua vnitate minor. Sit f eius valor pro $y=\mu$, i. e. quum habeatur $\psi\mu=\lambda m$, ſit

$$f=1-\frac{\lambda m}{\mu\psi'\mu} \text{ ſiue } \psi'\mu=\frac{\lambda m}{(1-f)\mu}$$

His ita praeparatis, conſideremus functionem ipſius y hanc

$$\frac{\lambda m}{(1-f)\mu}(y-\mu f)$$

quam ſtatuemus $=Fy$, nec non $dFy=F'y.dy$. Perſpicuum eſt fieri

$$F\mu = \lambda m = \psi\mu$$

$$F'\mu = \frac{\lambda\mu}{(1-f)\mu} = \psi'\mu$$

Quare quum $\psi'y$, aucta ipsa y, continuo crescat (saltem non decrescat, quod semper subintelligendum), $F'y$ vero constans sit, differentia $\psi'y - F'y = \frac{d(\psi y - Fy)}{dy}$ erit positiua pro valoribus ipsius y maioribus quam μ, negatiua pro minoribus. Hinc facile colligitur, $\psi y - Fy$ semper esse quantitatem positiuam, adeoque ψy semper erit absolute maior, saltem non minor, quam Fy, certe quamdiu valor ipsius Fy est positiuus, i. e. ab $y = \mu f$ vsque ad $y = 1$. Hinc valor integralis $\int (Fy)^2 dy$ ab $y = \mu f$ vsque ad $y = 1$ erit minor valore integralis $\int (\psi y)^2 dy$ inter eosdem limites, adeoque a potiori minor valore huius integralis ab $y = 0$ vsque ad $y = 1$, qui fit $= mm$. At valor integralis prioris inuenitur

$$= \frac{\lambda\lambda mm(1-\mu f)^3}{3\mu\mu(1-f)^2}$$

vnde colligimus, $\lambda\lambda$ esse minorem quam $\frac{3\mu\mu(1-f)^2}{(1-\mu f)^3}$, vbi f est quantitas inter 0 et 1 iacens. Iam valor fractionis $\frac{3\mu\mu(1-f)^2}{(1-\mu f)^3}$, cuius differentiale, si $= f$ tamquam quantitas variabilis consideratur, fit =

$$-\frac{3\mu\mu(1-f)}{(1-\mu f)^4} \cdot (2 - 3\mu + \mu f)\, df$$

continuo decrescit, dum f a valore 0 vsque ad valorem 1 transit, quoties μ minor est quam $\frac{2}{3}$, adeoque valor maximus possibilis erit is qui valori $f = 0$ respondet, puta $= 3\mu\mu$, ita vt in hoc casu λ certo fiat minor vel non maior quam $\mu\sqrt{3}$. *Q. E. P.* Contra quoties μ maior est quam $\frac{2}{3}$, valor istius fractionis erit maximus pro $2 - 3\mu + \mu f = 0$, i. e. pro $f = 3 - \frac{2}{\mu}$, vnde ille fit

$= \frac{4}{9(1-\mu)}$, adeoque in hoc casu λ non maior quam $\frac{2}{3\sqrt{(1-\mu)}}$. *Q. E. S.*

Ita e. g. pro $\mu = \frac{1}{2}$ certo λ nequit esse maior quam $\sqrt{\frac{3}{4}}$, i. e. error probabilis superare nequit limitem $0{,}8660254\,m$, cui in exemplo primo art. 9. aequalis inuentus est. Porro facile e theoremate nostro concluditur, μ non esse minorem quam $\lambda\sqrt{\frac{1}{3}}$, quamdiu λ minor sit quam $\sqrt{\frac{3}{4}}$, contra μ non esse minorem quam $1 - \frac{4}{9\lambda\lambda}$, pro valore ipsius λ maiori quam $\sqrt{\frac{4}{3}}$.

II.

Quum plura problemata infra tractanda etiam cum valore integralis $\int x^4 \varphi x . dx$ nexa sint, operae pretium erit, eum pro quibusdam casibus specialibus euoluere. Denotabimus valorem huius integralis ab $x = -\infty$ vsque ad $x = +\infty$ extensi per n^4.

I. Pro $\varphi x = \frac{1}{2a}$, quatenus x inter $-a$ et $+a$ continetur, habemus $n^4 = \frac{2}{5}a^4 = \frac{2}{5}m^4$.

II. In casu secundo art. 6, vbi $\varphi x = \frac{a \mp x}{aa}$, pro valoribus ipsius x inter o et $\pm a$, fit $n^4 = \frac{1}{15}a^4 = \frac{12}{5}m^4$.

III. In casu tertio vbi

$$\varphi x = \frac{e^{-\frac{xx}{hh}}}{a\sqrt{\pi}}$$

inuenitur per ea, quae in commentatione supra citata exponuntur, $n^4 = \frac{3}{4}h^4 = 3m^4$.

Ceterum demonstrari potest, valorem ipsius $\frac{n^4}{m^4}$ certo non esse minorem quam $\frac{2}{5}$, si modo suppositio art. praec. locum habeat.

12.

Denotantibus x, x', x'' etc. indefinite errores obseruationum eiusdem generis ab inuicem independentes, quorum probabilitates relatiuas exprimit praefixa characteristica φ; nec non y functionem datam rationalem indeterminatarum x, x', x'' etc.: integrale multiplex (I)

$$\int \varphi x . \varphi' x . \varphi'' x \ldots . dx . dx' . dx'' \ldots .$$

extenſum per omnes valores indeterminatarum x, x', x'', pro quibus valor ipſius y cadit intra limites datos o et η, exprimet probabilitatem valoris ipſius y indefinite intra o et η ſiti. Manifeſto hoc integrale erit functio ipſius η, cuius differentiale ſtatuemus $= \psi \eta . d\eta$, ita vt integrale ipſum fiat aequale integrali $\int \psi \eta . d\eta$ ab $\eta = 0$ incepto. Hoc pacto ſimul characteriſtica $\psi \eta$ probabilitatem relatiuam cuiusuis valoris ipſius y exprimere cenſenda eſt. Quum x conſiderari poſſit tamquam functio indeterminatarum y, x', x'' etc., quam ſtatuemus

$$= f(y, x', x'' \ldots .)$$

integrale (I) fiet

$$= \int \varphi . f(y, x', x'' \ldots .) . \frac{d f(y, x', x'' \ldots .)}{dy} . \varphi x' . \varphi x'' \ldots . dy . dx' . dx'' \ldots .$$

vbi y extendi debet ab $y = 0$ vsque ad $y = \eta$, indeterminatae reliquae vero per omnes valores, quibus refpondet valor realis ipſius $f(y, x', x'' \ldots .)$. Hinc colligitur

$$\psi y = \int \varphi . f(y, x', x'' \ldots .) . \frac{d f(y, x', x'' \ldots .)}{dy} . \varphi x' . \varphi x'' \ldots . dx' . dx'' \ldots .$$

integratione, in qua y tamquam conſtans conſiderari debet, extenſa per omnes valores indeterminatarum x', x'' etc., qui ipſi $f(y, x', x'' \ldots .)$ valorem realem conciliant.

13.

Ad hanc integrationem reipſa exſequendam cognitio functionis φ requireretur, quae plerumque incognita eſt: quinadeo,

etiamsi haec functio cognita esset, in plerisque casibus integratio vires analyseos superaret. Quae quum ita sint, probabilitatem quidem singulorum valorum ipsius y assignare non poterimus; at secus res se habebit, si tantummodo desideratur valor medius ipsius y, qui oritur ex integratione $\int y \psi y . dy$ per omnes valores ipsius y, quos quidem assequi potest, extensa. Et quum manifesto pro omnibus valoribus, quos y assequi nequit, vel per naturam functionis quam exprimit (e. g. pro negativis, si esset $y = xx + x'x' + x''x'' +$ etc.), vel ideo quod erroribus ipsis x, x', x'' etc. certi limites sunt positi, statuere oporteat $\psi y = 0$, manifesto res perinde se habebit, si integratio illa extendatur per omnes valores reales ipsius y, puta ab $y = -\infty$ vsque ad $y = +\infty$. Iam integrale $\int y \psi y . dy$ inter limites determinatos, puta ab $y = \eta$ vsque ad $y = \eta'$ sumtum aequale est integrali

$$\int y \varphi . f(y, x', x'' \ldots) \frac{df(y, x', x'' \ldots)}{dy} . \varphi x' . \varphi x'' \ldots dy . dx' . dx'' \ldots$$

integratione extensa ab $y = \eta$ vsque ad $y = \eta'$, atque per omnes valores indeterminatarum x', x'' etc. quibus respondet valor realis ipsius $f(y, x', x'' \ldots)$, siue quod idem est, valori integralis

$$\int y \varphi x . \varphi x' . \varphi x'' \ldots dx . dx' . dx'' \ldots$$

adhibendo in hac integratione pro y eius valorem per x, x', x'' etc. expressum, extendendoque eam per omnes harum indeterminatarum valores, quibus respondet valor ipsius y inter η et η' situs. Hinc colligimus, integrale $\int y \psi y . dy$ per omnes valores ipsius y, ab $y = -\infty$ vsque ad $y = +\infty$ extensum obtineri ex integratione

$$\int y \varphi x . \varphi x' . \varphi x'' \ldots dx . dx' . dx'' \ldots$$

per omnes valores reales ipsarum x, x', x'' etc. extensa, puta ab $x = -\infty$ vsque ad $x = +\infty$, ab $x' = -\infty$ vsque ad $x' = +\infty$ etc.

14.

Reducta itaque functione y ad formam aggregati talium partium

$A x^{\alpha} x'^{\beta} x''^{\gamma} \ldots$

valor integralis $\int y \psi y . dy$ per omnes valores ipſius y extenſi, ſeu valor medius ipſius y, aequalis erit aggregato partium

$$A \times \int x^{\alpha} \varphi x . dx \times \int x'^{\beta} \varphi x' . dx' \times \int x''^{\gamma} \varphi x'' . dx'' \ldots.$$

vbi integrationes extendendae ſunt ab $x = -\infty$ vsque ad $x = +\infty$, ab $x' = -\infty$ vsque ad $x' = +\infty$ etc.; ſiue quod eodem redit, aggregato partium quae oriuntur, dum pro ſingulis poteſtatibus x^{α}, x'^{β}, x''^{γ} etc. ipſarum valores medii ſubſtituuntur, cuius theorematis grauiſſimi veritas etiam ex aliis conſiderationibus facile deriuari potuiſſet.

15.

Applicemus ea, quae in art. praec. expoſuimus, ad caſum ſpecialem, vbi

$$y = \frac{xx + x'x' + x''x'' + \text{etc.}}{\sigma}$$

denotante σ multitudinem partium in numeratore. Valor medius ipſius y hic illico inuenitur $= mm$, accipiendo characterem m in eadem ſignificatione vt ſupra. Valor verus quidem ipſius y in caſu determinato maior minorue euadere poteſt medio, perinde ac valor verus termini ſimplicis xx: ſed probabilitas quod valor fortuitus ipſius y a medio mm haud ſenſibiliter aberret, continuo magis ad certitudinem appropinquabit creſcente multitudine σ. Quod quo clarius eluceat, quum probabilitatem ipſam exacte determinare non ſit in poteſtate, inueſtigemus errorem medium metuendum, dum ſupponimus $y = mm$. Manifeſto per principia art. 6. hic error erit radix quadrata valoris medii functionis

$$\left(\frac{xx + x'x' + x''x'' + \text{etc.}}{\sigma} - mm\right)^2$$

ad quem eruendum ſufficit obſeruare, valorem medium termini talis $\frac{x^4}{\sigma\sigma}$ eſſe $= \frac{n^4}{\sigma\sigma}$ (vtendo charactere n in ſignificatione art. 11.), valorem medium autem termini talis $\frac{2xxx'x'}{\sigma\sigma}$ fieri $= \frac{2m^4}{\sigma\sigma}$, vnde

facillime deducitur valor medius istius functionis

$$= \frac{n^4 - m^4}{\sigma}$$

Hinc discimus, si copia satis magna errorum fortuitorum ab inuicem independentium x, x', x' etc. in promtu sit, magna certitudine inde peti posse valorem approximatum ipsius m per formulam

$$m = \sqrt{\frac{(xx + x'x' + x''x'' + \text{etc.})}{\sigma}}$$

erroremque medium in hac determinatione metuendum, respectu quadrati mm, esse

$$= \sqrt{\frac{n^4 - m^4}{\sigma}}$$

Ceterum, quum posterior formula implicet quantitatem n, si id tantum agitur, vt idea qualiscunque de gradu praecisionis istius determinationis formari possit, sufficiet, aliquam hypothesin respectu functionis φ amplecti. E. g. in hypothesi tertia art. 9, 11. iste error fit $= mm\sqrt{\frac{2}{\sigma}}$. Quod si minus arridet, valor approximatus ipsius n^4 ex ipsis erroribus adiumento formulae

$$\frac{x^4 + x'^4 + x''^4 + \text{etc.}}{\sigma}$$

peti poterit. Generaliter autem affirmare possumus, praecisionem duplicatam in ista determinatione requirere errorum copiam quadruplicatam, siue pondus determinationis ipsi multitudini σ esse proportionale.

Prorsus simili modo, si obseruationum errores partem constantem inuoluunt, huius valor approximatus eo tutius e medio arithmetico multorum errorum colligi poterit, quo maior horum multitudo fuerit. Et quidem error medius in hac determinatione metuendus exprimetur per

$\sqrt{mm}$

$$\sqrt{\frac{mm - kk}{\sigma}}$$

si k designat partem constantem ipsam atque m errorem medium obseruationum parte constante nondum purgatarum, siue simpliciter per $\frac{m}{\sqrt{\sigma}}$, si m denotat errorem medium obseruationum a parte constante liberatarum (v. art. 8.).

16.

In artt. 12 — 15. supposuimus, errores x, x', x'' etc. ad idem obseruationum genus pertinere, ita vt singulorum probabilitates per eandem functionem exprimantur. Sed sponte patet, disquisitionem generalem artt. 12 — 14 aeque facile ad casum generaliorem extendi, vbi probabilitates errorum x, x', x'' etc. per functiones diuersas φx, $\varphi' x'$, $\varphi'' x''$ etc. exprimantur, i. e. vbi errores illi pertineant ad obseruationes praecisionis seu incertitudinis diuersae. Supponamus, x esse errorem obseruationis talis, cuius error medius metuendus sit $= m$; nec non x', x'' etc. esse errores aliarum obseruationum, quarum errores medii metuendi resp. sint m', m'' etc. Tunc valor medius aggregati $xx + x'x' + x''x'' +$ etc. erit $mm + m'm' + m''m'' +$ etc. Iam si aliunde constat, quantitates m, m', m'' etc. esse in ratione data, puta numeris 1, μ', μ'' etc. resp. proportionales, valor medius expressionis

$$\frac{xx + x'x' + x''x'' + \text{etc.}}{1 + \mu'\mu' + \mu''\mu'' + \text{etc.}}$$

erit $= mm$. Si vero valorem eiusdem expressionis determinatum, prout fors errores x, x', x'' etc. offert, ipsi mm aequalem ponimus, error medius, cui haec determinatio obnoxia manet, simili ratione vt in art. praec. inuenitur

$$= \frac{\sqrt{(n^4 + n'^4 + n''^4 + \text{etc.} - m^4 - m'^4 - m''^4 - \text{etc.})}}{1 + \mu'\mu' + \mu''\mu'' + \text{etc.}}$$

vbi n', n'' etc. respectu obseruationum, ad quas pertinent errores

x', x'' etc., idem denotare ſupponuntur, atque n reſpectu obſervationis primae. Quodſi itaque numeros n, n', n'' etc. ipſis m, m', m'' etc. proportionales ſupponere licet, error ille metuendus medius fit

$$= \frac{\sqrt{(n^4 - m^4)}.\sqrt{(1 + \mu'^4 + \mu''^4 + \text{etc.})}}{1 + \mu'\mu' + \mu''\mu'' + \text{etc.}}$$

At haecce ratio, valorem approximatum ipſius m determinandi, non eſt ea, quae maxime ad rem facit. Quod quo clarius oſtendamus, conſideremus expreſſionem generaliorem

$$y = \frac{xx + \alpha' x' x' + \alpha'' x'' x'' + \text{etc.}}{1 + \alpha'\mu'\mu' + \alpha''\mu''\mu'' + \text{etc.}}$$

cuius valor medius quoque erit $= mm$, quomodocunque eligantur coëfficientes α', α'' etc. Error autem medius metuendus, dum valorem determinatum ipſius y, prout fors errores x, x', x'' etc. offert, ipſi mm aequalem ſupponimus, inuenitur per principia ſupra tradita

$$= \frac{\sqrt{(n^4 - m^4 + \alpha'\alpha'(n'^4 - m'^4) + \alpha''\alpha''(n''^4 - m''^4) + \text{etc.})}}{1 + \alpha'\mu'\mu' + \alpha''\mu''\mu'' + \text{etc.}}$$

Vt hic error medius fiat quam minimus, ſtatuere oportebit

$$\alpha' = \frac{n^4 - m^4}{n'^4 - m'^4} \cdot \mu'\mu'$$

$$\alpha'' = \frac{n^4 - m^4}{n''^4 - m''^4} \cdot \mu''\mu'' \text{ etc.}$$

Manifeſto hi valores euolui nequeunt, niſi inſuper ratio quantitatum n, n', n'' etc. ad m, m', m'' etc aliunde nota fuerit; qua cognitione exacta deficiente, ſaltem tutiſſimum videtur *), illas his proportionales ſupponere (v. art. 11), vnde prodeunt valores

*) Scilicet cognitionem quantitatum μ', μ'' etc. in eo ſolo caſu in potestate eſſe concipimus, vbi per rei naturam errores x, x', x'' etc. ipſis 1, μ', μ'' etc. proportionales, aeque probabiles cenſendi ſunt, aut potius vbi

$$\varphi x = \mu'\varphi'(\mu' x) = \mu''\varphi''(\mu'' x) \text{ etc.}$$

$$\alpha' = \frac{1}{\mu'\mu'}, \quad \alpha'' = \frac{1}{\mu''\mu''} \text{ etc.}$$

i. e. coëfficientes α', α'' etc. aequales ſtatui debent ponderibus relatiuis obſeruationum, ad quas pertinent errores x', x'' etc., aſſumto pondere obſeruationis, ad quam pertinet error x, pro vnitate. Hoc pacto, deſignante vt ſupra σ multitudinem errorum propoſitorum, habebimus valorem medium expreſſionis

$$\frac{xx + \alpha' x' x' + \alpha'' x'' x'' + \text{etc.}}{\sigma}$$

$= mm$, atque errorem medium metuendum, dum valorem fortuito determinatum huius expreſſionis pro valore vero ipſius mm adoptamus

$$\frac{\sqrt{(n^4 + \alpha'\alpha' n'^4 + \alpha''\alpha'' n''^4 + \text{etc.} - \sigma m^4)}}{\sigma}$$

et proin, ſiquidem licet, ipſas n, n', n'' etc. ipſis m, m', m'' proportionales ſupponere,

$$= \sqrt{\frac{n^4 - m^4}{\sigma}}$$

quae formula identica eſt cum ea, quam ſupra pro caſu obſeruationum eiusdem generis inueneramus.

17.

Si valor quantitatis, quae ab alia quantitate incognita pendet, per obſeruationem praeciſione abſoluta non gaudentem determinata eſt, valor incognitae hinc calculatus etiam errori obnoxius erit, ſed nihil in hac determinatione arbitrio relinquitur. At ſi *plures* quantitates ab eadem incognita pendentes per obſeruationes haud abſolute exactas iunotuerunt, valorem incognitae vel per quamlibet harum obſeruationum eruere poſſumus, vel per aliquam plurium obſeruationum combinationem, quod infinitis modis diuerſis fieri poteſt. Quamquam vero valor incognitae tali modo prodiens errori ſemper obnoxius manet, tamen in alia

combinatione maior, in alia minor error metuendus erit. Similiter res se habebit, si plures quantitates a pluribus incognitis simul pendentes sunt obseruatae: prout obseruationum multitudo multitudini incognitarum vel aequalis, vel hac minor vel maior fuerit, problema vel determinatum, vel indeterminatum, vel plus quam determinatum erit (generaliter saltem loquendo), et in casu tertio ad incognitarum determinationem obseruationes infinitis modis diuersis combinari poterunt. E tali combinationum varietate eas eligere, quae maxime ad rem faciant, i. e. quae incognitarum valores erroribus minimis obnoxios suppeditent, problema sane est in applicatione matheseos ad philosophiam naturalem longe grauissimum.

In Theoria motus corporum coelestium ostendimus, quomodo valores incognitarum *maxime probabiles* eruendi sint, si lex probabilitatis errorum obseruationum cognita sit; et quum haec lex natura sua in omnibus fere casibus hypothetica maneat, theoriam illam ad legem maxime plausibilem applicauimus, vbi probabilitas erroris x quantitati exponentiali e^{-hhxx} proportionalis supponitur, vnde methodus a nobis dudum in calculis praesertim astronomicis, et nunc quidem a plerisque calculatoribus sub nomine methodi quadratorum minimorum vsitata demanauit.

Postea ill. Laplace, rem alio modo aggressus, idem principium omnibus aliis etiamnum praeferendum esse docuit, quaecunque fuerit lex probabilitatis errorum, si modo obseruationum multitudo sit permagna. At pro multitudine obseruationum modica, res intacta mansit, ita vt si lex nostra hypothetica respuatur, methodus quadratorum minimorum eo tantum nomine prae aliis commendabilis habenda sit, quod calculorum concinnitati maxime est adaptata.

Geometris itaque gratum fore speramus, si in hac noua argumenti tractatione docuerimus, methodum quadratorum mini-

morum exhibere combinationem ex omnibus optimam, non quidem proxime, ſed abſolute, quaecunque fuerit lex probabilitatis errorum, quaecunque obſeruationum multitudo, ſi modo notionem erroris medii non ad mentem ill. Laplace, ſed ita vt in artt. 5 et 6 a nobis factum eſt ſtabiliamus.

Ceterum expreſſis verbis hic praemonere conuenit, in omnibus diſquiſitionibus ſequentibus tantummodo de erroribus irregularibus atque a parte conſtante liberis ſermonem eſſe, quum proprie ad perfectam artem obſeruandi pertineat, omnes errorum conſtantium cauſſas ſummo ſtudio amouere. Quaenam vero ſubſidia calculator tales obſeruationes tractare ſuſcipiens, quas ab erroribus conſtantibus non liberas eſſe iuſta ſuſpicio adeſt, ex ipſo calculo probabilium petere poſſit, diſquiſitioni peculiari alia occaſione promulgandae reſeruamus.

18.

PROBLEMA.

Deſignante U functionem datam quantitatum incognitarum V, V', V'' etc., quaeritur error medius M in determinatione valoris ipſius U metuendus, ſi pro V, V', V'' etc. adoptentur non valores veri, ſed ii, qui ex obſeruationibus ab invicem independentibus, erroribus mediis m, m', m'' etc. reſp. obnoxiis prodeunt.

Sol. Denotatis erroribus in valoribus obſeruatis ipſarum V, V', V'' etc. per e, e', e'' etc., error inde redundans in valorem ipſius U exprimi poterit per functionem linearem

$$\lambda e + \lambda' e' + \lambda'' e'' + \text{etc.} = E$$

vbi λ, λ', λ'' etc. ſunt valores quotientium differentialium $\frac{dU}{dV}$, $\frac{dU}{dV'}$, $\frac{dU}{dV''}$ etc. pro valoribus veris ipſarum V, V', V'' etc., ſiquidem obſeruationes ſatis exactae ſunt, vt errorum quadrata pro-

ductaque negligere liceat. Hinc primo ſequitur, quoniam obſervationum errores a partibus conſtantibus liberi ſupponuntur, valorem medium ipſius E eſſe $= 0$. Porro error medius in valore ipſius U metuendus, erit radix quadrata e valore medio ipſius EE, ſiue MM erit valor medius aggregati

$$\lambda\lambda ee + \lambda'\lambda' e'e' + \lambda''\lambda'' e''e'' + \text{etc.} + 2\lambda\lambda' ee' + 2\lambda\lambda'' ee'' + 2\lambda'\lambda'' e'e'' + \text{etc.}$$

At valor medius ipſius $\lambda\lambda ee$ fit $\lambda\lambda mm$, valor medius ipſius $\lambda'\lambda' e'e'$ fit $= \lambda'\lambda' m'm'$ etc.; denique valores medii productorum $2\lambda\lambda' ee'$ etc. omnes fiunt $= 0$ Hinc itaque colligimus

$$M = \sqrt{(\lambda\lambda mm + \lambda'\lambda' m'm' + \lambda''\lambda'' m''m'' + \text{etc.})}$$

Huic ſolutioni quasdam annotationes adiicere conueniet.

I. Quatenus ſpectando obſeruationum errores tanquam quantitates primi ordinis, quantitates ordinum altiorum negliguntur, in formula noſtra pro λ, λ' λ'' etc. etiam valores eos quotientium $\frac{dU}{dV}$ etc. adoptare licebit, qui prodeunt e valoribus obſervatis quantitatum V, V', V'' etc. Quoties U eſt functio linearis, manifeſto nulla prorſus erit differentia.

II. Si loco errorum mediorum obſeruationum, harum pondera introducere malumus, ſint haec, ſecundum vnitatem arbitrariam, reſp p, p', p'' etc., atque P pondus determinationis valoris ipſius U e valoribus obſeruatis quantitatum V, V', V'' etc. prodeuntis. Ita habebimus

$$P = \frac{1}{\frac{\lambda\lambda}{p} + \frac{\lambda'\lambda'}{p'} + \frac{\lambda''\lambda''}{p''} + \text{etc.}}$$

III. Si T eſt functio alia data quantitatum V, V', V'' etc. atque, pro harum valoribus veris,

$$\frac{dT}{dV} = \varkappa, \quad \frac{dT}{dV'} = \varkappa', \quad \frac{dT}{dV''} = \varkappa'' \text{ etc.}$$

error in determinatione valoris ipsius T, è valoribus obseruatis ipsarum V, V', V'' etc. petita, erit $= \varkappa e + \varkappa' e' + \varkappa'' e'' +$ etc., $= E'$, atque error medius in ista determinatione metuendus $= \sqrt{(\varkappa\varkappa mm + \varkappa'\varkappa' m'm' + \varkappa''\varkappa'' m''m'' + \text{etc.})}$. Errores E, E' vero manifesto ab inuicem iam non erunt independentes, valorque medius producti EE', secus ac valor medius producti ee', non erit $= 0$, sed $= \varkappa\lambda mm + \varkappa'\lambda' m'm' + \varkappa''\lambda'' m''m'' +$ etc.

IV. Problema nostrum etiam ad casum eum extendere licet, vbi valores quantitatum V, V', V'' etc. non immediate per obseruationes inueniuntur, sed quomodocunque ex obseruationum combinationibus deriuantur, si modo singularum determinationes ab inuicem sunt independentes, i. e. obseruationibus diuersis superstructae: quoties autem haec conditio locum non habet, formula pro M erronea euaderet. E. g. si vna alteraue obseruatio, quae ad determinationem valoris ipsius V inseruiit, etiam ad valorem ipsius V' determinandum adhibita esset, errores e et e' haud amplius ab inuicem independentes forent, neque adeo producti ee' valor medius $= 0$. Si vero in tali casu nexus quantitatum V, V' cum obseruationibus simplicibus, e quibus deductae sunt, rite perpenditur, valor medius producti ee' adiumento annotationis III. assignari, atque sic formula pro M completa reddi poterit.

19.

Sint V, V', V'' etc. functiones incognitarum x, y, z etc., multitudo illarum $= \pi$, multitudo incognitarum $= \rho$, supponamusque, per obseruationes vel immediate vel mediate valores functionum inuentos esse $V = L$, $V' = L'$, $V'' = L''$ etc., ita tamen vt hae determinationes ab inuicem fuerint independentes. Si ρ maior est quam π, incognitarum euolutio manifesto fit problema indeterminatum; si ρ ipsi π aequalis est, singulae x, y, z etc. in formam functionum ipsarum V, V', V'' etc. redigi vel redactae

concipi possunt, ita vt ex harum valoribus obseruatis valores istarum inueniri possint, simulque adiumento art. praec. praecisionem relatiuam singulis his determinationibus tribuendam assignare liceat; denique si ρ minor est quam π, singulae x, y, z etc. infinitis modis diuersis in formam functionum ipsarum V, V' V'' etc. redigi, adeoque illarum valores infinitis modis diuersis erui poterunt. Quae determinationes exacte quidem quadrare deberent, si obseruationes praecisione absoluta gauderent; quod quum secus se habeat, alii modi alios valores suppeditabunt, nec minus determinationes e combinationibus diuersis petitae inaequali praecisione instructae erunt.

Ceterum si in casu secundo vel tertio functiones V, V', V'' etc. ita comparatae essent, vt $\pi - \rho + 1$ ex ipsis, vel plures, tamquam functiones reliquarum spectare liceret, problema respectu posteriorum functionum etiamnum plus quam determinatum esset, respectu incognitarum x, y, z etc. autem indeterminatum; harum scilicet valores ne tunc quidem determinare liceret, quando valores functionum V, V', V'' etc. absolute exacti dati essent: sed hunc casum a disquisitione nostra excludemus.

Quoties V, V', V'' etc. per se non sunt functiones *lineares* indeterminatarum suarum, hoc efficietur, si loco incognitarum primitiuarum introducuntur ipsarum differentiae a valoribus approximatis, quos aliunde cognitos esse supponere licet. Errores medios in determinationibus $V = L$, $V' = L'$, $V'' = L''$ etc. metuendos resp. denotabimus per m, m', m'' etc., determinationumque pondera per p, p', p'' etc., ita vt sit $pmm = p'm'm' = p''m''m''$ etc. Rationem, quam inter se tenent errores medii, cognitam supponemus, ita vt pondera, quorum vnum ad lubitum accipi potest, sint nota. Denique statuemus

$$(V - L)\sqrt{p} = v,\ (V' - L')\sqrt{p'} = v',\ (V'' - L'')\sqrt{p''} = v'' \text{ etc.}$$

Manifesto itaque res perinde se habebit, ac si obseruationes immedia-

mediatae, aequali praecisione gaudentes, puta quarum error medius $= m\sqrt{p} = m'\sqrt{p'} = m''\sqrt{p''}$ etc., siue quibus pondus $= 1$ tribuitur, suppeditauissent

$$v = 0,\ v' = 0,\ v'' = 0 \text{ etc.}$$

20.

PROBLEMA.

Designantibus v, v', v'' etc. functiones lineares indeterminatarum x, y, z etc. sequentes

$$\left.\begin{aligned} v &= ax + by + cz + \text{etc.} + l \\ v' &= a'x + b'y + c'z + \text{etc.} + l' \\ v'' &= a''x + b''y + c''z + \text{etc.} + l'' \text{ etc.} \end{aligned}\right\} \text{(I)}$$

ex omnibus systematibus coëfficientium $\varkappa$, $\varkappa'$, $\varkappa''$ etc., qui indefinite dant

$$\varkappa v + \varkappa' v' + \varkappa'' v'' + \text{etc.} = x - k$$

ita vt k sit quantitas determinata i. e. ab x, y, z etc. independens, eruere id, pro quo $\varkappa\varkappa + \varkappa'\varkappa' + \varkappa''\varkappa'' + \text{etc.}$ nanciscatur valorem minimum.

Solutio. Statuamus

$$\left.\begin{aligned} av + a'v' + a''v'' + \text{etc.} &= \xi \\ bv + b'v' + b''v'' + \text{etc.} &= \eta \\ cv + c'v' + c''v'' + \text{etc.} &= \zeta \end{aligned}\right\} \text{(II)}$$

etc.: eruntque etiam ξ, η, ζ etc. functiones lineares ipsarum x, y, z etc., puta

$$\left.\begin{aligned} \xi &= x\Sigma aa + y\Sigma ab + z\Sigma ac + \text{etc.} + \Sigma al \\ \eta &= x\Sigma ab + y\Sigma bb + z\Sigma bc + \text{etc.} + \Sigma bl \\ \zeta &= x\Sigma ac + y\Sigma bc + z\Sigma cc + \text{etc.} + \Sigma cl \text{ etc.} \end{aligned}\right\} \text{(III)}$$

(vbi Σaa denotat aggregatum $aa + a'a' + a''a'' + \text{etc.}$, ac perinde de reliquis),

multitudoque ipſarum ξ, η, ζ etc. multitudini indeterminatarum x, y, z etc. aequalis, puta $= \pi$. Per eliminationem itaque elici poterit aequatio talis *)

$$x = A + [\alpha\alpha]\xi + [\alpha\beta]\eta + [\alpha\gamma]\zeta + \text{etc.}$$

in qua ſubſtituendo pro ξ, η, ζ etc. valores earum ex III, aequatio identica prodire debet. Quare ſtatuendo

$$\left.\begin{array}{l} a[\alpha\alpha] + b[\alpha\beta] + c[\alpha\gamma] + \text{etc.} = \alpha \\ a'[\alpha\alpha] + b'[\alpha\beta] + c'[\alpha\gamma] + \text{etc.} = \alpha' \\ a''[\alpha\alpha] + b''[\alpha\beta] + c''[\alpha\gamma] + \text{etc.} = \alpha'' \text{ etc.} \end{array}\right\} \text{(IV)}$$

neceſſario erit indefinite

$$\alpha u + \alpha' v' + \alpha'' v'' + \text{etc.} = x - A \quad \text{(V)}$$

Haec aequatio docet, inter ſyſtemata valorum coëfficientium $\varkappa$, $\varkappa'$, $\varkappa''$ etc. certo etiam referendos eſſe hos $\varkappa = \alpha$, $\varkappa' = \alpha'$, $\varkappa'' = \alpha''$ etc., nec non, pro ſyſtemate quocunque, fieri debere indefinite

$$(\varkappa - \alpha)v + (\varkappa' - \alpha')v' + (\varkappa'' - \alpha'')v'' + \text{etc.} = A - k$$

quae aequatio implicat ſequentes

$$(\varkappa - \alpha)a + (\varkappa' - \alpha')a' + (\varkappa'' - \alpha'')a'' + \text{etc.} = 0$$
$$(\varkappa - \alpha)b + (\varkappa' - \alpha')b' + (\varkappa'' - \alpha'')b'' + \text{etc.} = 0$$
$$(\varkappa - \alpha)c + (\varkappa' - \alpha')c' + (\varkappa'' - \alpha'')c'' + \text{etc.} = 0 \text{ etc.}$$

Multiplicando has aequationes reſp. per $[\alpha\alpha]$, $[\alpha\beta]$, $[\alpha\gamma]$ etc., et addendo, obtinemus propter (IV)

$$(\varkappa - \alpha)\alpha + (\varkappa' - \alpha')\alpha' + (\varkappa'' - \alpha'')\alpha'' + \text{etc.} = 0$$

ſine quod idem eſt

$$\varkappa\varkappa + \varkappa'\varkappa' + \varkappa''\varkappa'' + \text{etc.} = \alpha\alpha + \alpha'\alpha' + \alpha''\alpha'' + \text{etc.}$$
$$+ (\varkappa - \alpha)^2 + (\varkappa' - \alpha')^2 + (\varkappa'' - \alpha'')^2 + \text{etc.}$$

vnde patet, aggregatum $\varkappa\varkappa + \varkappa'\varkappa' + \varkappa''\varkappa'' + \text{etc.}$ valorem minimum obtinere, ſi ſtatuatur $\varkappa = \alpha$, $\varkappa' = \alpha'$, $\varkappa'' = \alpha''$ etc. *Q. E. I.*

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*) Ratio, cur ad denotandos coëfficientes e tali eliminatione prodeuntes, hos potiſſimum characteres elegerimus, infra elucebit.

Ceterum hic valor minimus ipse sequenti modo eruitur. Aequatio (V) docet esse

$$\alpha a + \alpha' a' + \alpha'' a'' + \text{etc.} = 1$$
$$\alpha b + \alpha' b' + \alpha'' b'' + \text{etc.} = 0$$
$$\alpha c + \alpha' c' + \alpha'' c'' + \text{etc.} = 0 \text{ etc.}$$

Multiplicando has aequationes resp. per $[\alpha\alpha]$, $[\alpha\beta]$, $[\alpha\gamma]$ etc. et addendo, protinus habemus adiumento aequationum (IV)

$$\alpha\alpha + \alpha'\alpha' + \alpha''\alpha'' + \text{etc.} = [\alpha\alpha]$$

21.

Quum obseruationes suppeditauerint aequationes (proxime veras) $v = 0$, $v' = 0$, $v'' = 0$ etc., ad valorem incognitae x inde eliciendum, combinatio illarum aequationum talis

$$\varkappa v + \varkappa' v' + \varkappa'' v'' + \text{etc.} = 0$$

adhibenda est, quae ipsi x coëfficientem 1 conciliet, incognitasque reliquas y, z etc. eliminet; cui determinationi per art. 18. pondus

$$= \frac{1}{\varkappa\varkappa + \varkappa'\varkappa' + \varkappa''\varkappa'' + \text{etc.}}$$

tribuendum erit. Ex art. praec. itaque sequitur, determinationem maxime idoneam eam fore, vbi statuatur $\varkappa = \alpha$, $\varkappa' = \alpha'$, $\varkappa'' = \alpha''$ etc. Hoc pacto x obtinet valorem A, manifestoque idem valor etiam (absque cognitione multiplicatorum α, α', α'' etc.) protinus per eliminationem ex aequationibus $\xi = 0$, $\eta = 0$, $\zeta = 0$ etc. elici potest.

Pondus huic determinationi tribuendum erit $= \frac{1}{[\alpha\alpha]}$, siue error medius in ipsa metuendus

$$= m\sqrt{p}\,[\alpha\alpha] = m'\sqrt{p'}\,[\alpha\alpha] = m''\sqrt{p''}\,[\alpha\alpha] \text{ etc.}$$

Prorsus simili modo determinatio maxime idonea incognitarum reliquarum y, z etc. eosdem valores ipsis conciliabit, qui

per eliminationem ex iisdem aequationibus $\xi = 0$, $\eta = 0$, $\zeta = 0$ etc. prodeunt.

Denotando aggregatum indefinitum $vv + v'v' + v''v''$ etc., siue quod idem est hoc

$$p(V-L)^2 + p'(V'-L')^2 + p''(V''-L'')^2 + \text{etc.}$$

per Ω, patet, 2ξ, 2η, 2ζ etc. esse quotientes differentiales partiales functionis Ω, puta

$$2\xi = \frac{d\Omega}{dx}, \quad 2\eta = \frac{d\Omega}{dy}, \quad 2\zeta = \frac{d\Omega}{dz} \text{ etc.}$$

Quapropter valores incognitarum ex obseruationum combinatione maxime idonea prodeuntes, quos *valores maxime plausibiles* commode vocare possumus, identici erunt cum iis, per quos Ω valorem minimum obtinet. Iam $V - L$ indefinite exprimit differentiam inter valorem computatum et obseruatum. Valores itaque incognitarum maxime plausibiles iidem erunt, qui summam quadratorum differentiarum inter quantitatum V, V', V'' etc. valores obseruatos et computatos, per obseruationum pondera multiplicatorum, minimam efficiunt, quod principium in *Theoria Motus Corporum Coelestium* longe alia via stabilieueramus. Et si insuper praecisio relatiua singularum determinationum assignanda est, per eliminationem indefinitam ex aequationibus (III) ipsas x, y, z etc. in tali forma exhibere oportet:

$$\left.\begin{aligned} x &= A + [\alpha\alpha]\xi + [\alpha\beta]\eta + [\alpha\gamma]\zeta + \text{etc.} \\ y &= B + [\beta\alpha]\xi + [\beta\beta]\eta + [\beta\gamma]\zeta + \text{etc.} \\ z &= C + [\gamma\alpha]\xi + [\gamma\beta]\eta + [\gamma\gamma]\zeta + \text{etc.} \end{aligned}\right\} \text{(VII)}$$

etc.

quo pacto valores maxime plausibiles incognitarum x, y, z etc. erunt resp. A, B, C etc., atque pondera his determinationibus

tribuenda $\frac{1}{[\alpha\alpha]}$, $\frac{1}{[\beta\beta]}$, $\frac{1}{[\gamma\gamma]}$ etc., ſiue errores medii in ipſis metuendi

pro x.....$m\sqrt{p}[\alpha\alpha] = m'\sqrt{p'}[\alpha\alpha] = m''\sqrt{p''}[\alpha\alpha]$ etc.

pro y.....$m\sqrt{p}[\beta\beta] = m'\sqrt{p'}[\beta\beta] = m''\sqrt{p''}[\beta\beta]$ etc.

pro z.....$m\sqrt{p}[\gamma\gamma] = m'\sqrt{p'}[\gamma\gamma] = m''\sqrt{p''}[\gamma\gamma]$ etc.

etc.

quod conuenit cum iis, quae in *Theoria Motus Corporum Coeleſtium* docuimus.

22.

De caſu omnium ſimpliciſſimo, ſimul vero frequentiſſimo, vbi vnica incognita adeſt, atque $V = x$, $V' = x$, $V'' = x$ etc., paucis ſeorſim agere conueniet. Erit ſcilicet $a = \sqrt{p}$, $a' = \sqrt{p'}$, $a'' = \sqrt{p''}$ etc., $l = -L\sqrt{p}$, $l' = -L'\sqrt{p'}$, $l'' = -L''\sqrt{p''}$ etc., et proin

$$\xi = (p + p' + p'' + \text{etc.})\, x - (pL + p'L' + p''L'' + \text{etc.})$$

Hinc porro

$$[\alpha\alpha] = \frac{1}{p + p' + p'' + \text{etc.}}$$

$$A = \frac{pL + p'L' + p''L'' + \text{etc.}}{p + p' + p'' + \text{etc.}}$$

Si itaque e pluribus obſeruationibus inaequali praeciſione gaudentibus, et quarum pondera reſp. ſunt p, p', p'' etc., valor eiusdem quantitatis inuentus eſt e prima $= L$, e ſecuuda $= L'$, e tertia $= L''$ etc., huius valor maxime plauſibilis erit

$$= \frac{pL + p'L' + p''L'' + \text{etc.}}{p + p' + p'' + \text{etc.}}$$

pondusque huius determinationis $= p + p' + p''$ etc. Si omnes

obſeruationes aequali praeciſione gaudent, valor maxime plauſibilis erit

$$= \frac{L + L' + L'' + \text{ etc.}}{\pi}$$

i. e. aequalis medio arithmetico valorum obſeruatorum, huiusque determinationis pondus $= \pi$, accepto pondere obſeruationum pro vnitate.

THEORIA
COMBINATIONIS OBSERVATIONUM
ERRORIBUS MINIMIS OBNOXIAE.

PARS POSTERIOR.

SOCIETATI REGIAE EXHIBITA FEBR. 2, 1823.

23.

Plures adhuc ſuperſunt diſquiſitiones, per quas theoria praecedens tum illuſtrabitur tum ampliabitur.

Ante omnia inueſtigare oportet, num negotium eliminationis, cuius adiumento indeterminatae x, y, z etc. per ξ, η, ζ etc. exprimendae ſunt, ſemper ſit poſſibile. Quum multitudo illarum multitudini harum aequalis ſit, e theoria eliminationis in aequationibus linearibus conſtat, illam eliminationem, ſi ξ, η, ζ etc. ab inuicem independentes ſint, certo poſſibilem fore; ſin minus, impoſſibilem. Supponamus aliquantiſper, ξ, η, ζ etc. non eſſe ab inuicem independentes, ſed exſtare inter ipſas aequationem identicam

$$o = F\xi + G\eta + H\zeta + \text{etc.} + K$$

Habebimus itaque

$$F\Sigma aa + G\Sigma ab + H\Sigma ac + \text{etc.} = o$$
$$F\Sigma ab + G\Sigma bb + H\Sigma bc + \text{etc.} = o$$
$$F\Sigma ac + G\Sigma bc + H\Sigma cc + \text{etc.} = o$$

etc., nec non

$$F\Sigma al + G\Sigma bl + H\Sigma cl + \text{etc.} = -K$$

Statuendo porro

$$\left.\begin{array}{l} aF + bG + cH + \text{etc.} = \theta \\ a'F + b'G + c'H + \text{etc.} = \theta' \\ a''F + b''G + c''H + \text{etc.} = \theta'' \end{array}\right\} (\mathrm{I})$$

etc., eruitur

$$a\theta + a'\theta' + a''\theta'' + \text{etc.} = 0$$
$$b\theta + b'\theta' + b''\theta'' + \text{etc.} = 0$$
$$c\theta + c'\theta' + c''\theta'' + \text{etc.} = 0$$

etc., nec non

$$l\theta + l'\theta' + l''\theta'' + \text{etc.} = -K$$

Multiplicando itaque aequationes (I) resp. per θ, θ', θ'' etc. et addendo, obtinemus:

$$0 = \theta\theta + \theta'\theta' + \theta''\theta'' + \text{etc.}$$

quae aequatio manifesto consistere nequit, nisi simul fuerit $\theta = 0$, $\theta' = 0$, $\theta'' = 0$ etc. Hinc primo colligimus, necessario esse debere $K = 0$. Dein aequationes (I) docent, functiones v, v', v'' etc. ita comparatas esse, vt ipsarum valores non mutentur, si valores quantitatum x, y, z etc. capiant incrementa vel decrementa ipsis F, G, H etc. resp. proportionalia, idemque manifesto de functionibus V, V', V'' etc. valebit. Suppositio itaque consistere nequit, nisi in casu tali, vbi vel e valoribus exactis quantitatum V, V', V'' etc. valores incognitarum x, y, z etc. determinare impossibile fuisset, i. e. vbi problema natura sua fuisset indeterminatum, quem casum a disquisitione nostra exclusimus.

24.

Denotemus per β, β', β'' etc. multiplicatores, qui eandem relationem habent ad indeterminatam y, quam habent α, α', α'' etc. ad x, puta sit

$a[\alpha\beta]$

$$a[\beta\alpha] + b[\beta\beta] + c[\beta\gamma] + \text{etc.} = \beta$$
$$a'[\beta\alpha] + b'[\beta\beta] + c'[\beta\gamma] + \text{etc.} = \beta'$$
$$a''[\beta\alpha] + b''[\beta\beta] + c''[\beta\gamma] + \text{etc.} = \beta''$$

etc., ita vt fiat indefinite

$$\beta v + \beta' v' + \beta'' v'' + \text{etc.} = y - B$$

Perinde ſint γ, γ', γ'' etc. multiplicatores ſimiles reſpectu indeterminatae z, puta

$$a[\gamma\alpha] + b[\gamma\beta] + c[\gamma\gamma] + \text{etc.} = \gamma$$
$$a'[\gamma\alpha] + b'[\gamma\beta] + c'[\gamma\gamma] + \text{etc.} = \gamma'$$
$$a''[\gamma\alpha] + b''[\gamma\beta] + c''[\gamma\gamma] + \text{etc.} = \gamma''$$

etc., ita vt fiat indefinite

$$\gamma v + \gamma' v' + \gamma'' v'' + \text{etc.} = z - C$$

et ſic porro. Hoc pacto, perinde vt iam in art. 20. inueneramus

$$\Sigma\alpha a = 1,\ \Sigma\alpha b = 0,\ \Sigma\alpha c = 0,\ \text{etc., nec non } \Sigma\alpha l = -A,$$

etiam habebimus

$$\Sigma\beta a = 0,\ \Sigma\beta b = 1,\ \Sigma\beta c = 0 \text{ etc., atque } \Sigma\beta l = -B$$
$$\Sigma\gamma a = 0,\ \Sigma\gamma b = 0,\ \Sigma\gamma c = 1 \text{ etc., atque } \Sigma\gamma l = -C$$

et ſic porro. Nec minus, quemadmodum in art. 20. prodiit $\Sigma\alpha\alpha = [\alpha\alpha]$, etiam erit

$$\Sigma\beta\beta = [\beta\beta],\ \Sigma\gamma\gamma = [\gamma\gamma] \text{ etc.}$$

Multiplicando porro valores ipſorum α, α', α'' etc. (art. 20. IV) reſp. per β, β', β'' etc. et addendo, obtinemus

$$\alpha\beta + \alpha'\beta' + \alpha''\beta'' \text{ etc.} = [\alpha\beta], \text{ ſiue } \Sigma\alpha\beta = [\alpha\beta].$$

Multiplicando autem valores ipſorum β, β', β'' etc. reſp. per α, α', α'' etc., et addendo, perinde prodit

$$\alpha\beta + \alpha'\beta' + \alpha''\beta'' + \text{etc.} = [\beta\alpha], \text{ adeoque } [\alpha\beta] = [\beta\alpha]$$

Prorſus ſimili modo eruitur

$$[\alpha\gamma] = [\gamma\alpha] = \Sigma\alpha\gamma,\ [\beta\gamma] = [\gamma\beta] = \Sigma\beta\gamma \text{ etc.}$$

25.

Denotemus porro per λ, λ', λ'' etc. valores functionum v, v', v'' etc., qui prodeunt, dum pro x, y, z etc. ipſarum valores maxime plauſibiles A, B, C etc. ſubſtituuntur, puta

$$aA + bB + cC + \text{etc.} + l = \lambda$$
$$a'A + b'B + c'C + \text{etc.} + l' = \lambda'$$
$$a''A + b''B + c''C + \text{etc.} + l'' = \lambda''$$

etc.; ſtatuamus praeterea

$$\lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' + \text{etc.} = M$$

ita vt ſit M valor functionis Ω valoribus maxime plauſibilibus indeterminatarum reſpondens, adeoque per ea, quae in art. 20. demonſtrauimus, valor minimus huius functionis. Hinc erit $a\lambda + a'\lambda' + a''\lambda'' +$ etc. valor ipſius ξ, valoribus $x = A$, $y = B$, $z = C$ etc. reſpondens, adeoque $= 0$, i. e. habebimus

$$\Sigma a\lambda = 0$$

et perinde fiet

$$\Sigma b\lambda = 0,\ \Sigma c\lambda = 0 \text{ etc.; nec non } \Sigma\alpha\lambda = 0,\ \Sigma\beta\lambda = 0,$$
$$\Sigma\gamma\lambda = 0 \text{ etc.}$$

Denique multiplicando expreſſiones ipſarum $\lambda, \lambda', \lambda''$ etc. per $\lambda, \lambda', \lambda''$ etc. reſp., et addendo, obtinemus $l\lambda + l'\lambda' + l''\lambda'' +$ etc. $= \lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' +$ etc., ſiue

$$\Sigma l\lambda = M.$$

26.

Substituendo in aequatione $v = ax + by + cz +$ etc. $+ l$, pro x, y, z etc. expreſſiones VII. art. 21, prodibit, adhibitis reductionibus ex praecedentibus obuiis,

$$v = \alpha\xi + \beta\eta + \gamma\zeta + \text{etc.} + \lambda$$

et perinde erit indefinite

$$v' = \alpha'\xi + \beta'\eta + \gamma'\zeta + \text{etc.} + \lambda'$$
$$v'' = \alpha''\xi + \beta''\eta + \gamma''\zeta + \text{etc.} + \lambda''$$

etc. Multiplicando vel has aequationes, vel aequationes I art. 20. reſp. per $\lambda, \lambda', \lambda''$ etc., et addendo, diſcimus eſſe indefinite

$$\lambda v + \lambda' v' + \lambda'' v'' + \text{etc.} = M.$$

27.

Functio Ω indefinite in pluribus formis exhiberi poteſt, quas euoluere operae pretium erit. Ac primo quidem quadrando aequationes I art. 20. et addendo, ſtatim fit

$$\Omega = xx\Sigma aa + yy\Sigma bb + zz\Sigma cc + \text{etc.} + 2xy\Sigma ab + 2xz\Sigma ac + 2yz\Sigma bc + \text{etc.} + 2x\Sigma al + 2y\Sigma bl + 2z\Sigma cl + \text{etc.} + \Sigma ll.$$

quae eſt forma *prima*.

Multiplicando easdem aequationes reſp. per v, v', v'' etc., et addendo, obtinemus:

$$\Omega = \xi x + \eta y + \zeta z + \text{etc.} + lv + l'v' + l''v'' + \text{etc.}$$

atque hinc, ſubſtituendo pro v, v', v'' etc. expreſſiones in art. praec. traditas,

$$\Omega = \xi x + \eta y + \zeta z + \text{etc.} - A\xi - B\eta - C\zeta - \text{etc.} + M$$

ſiue

$$\Omega = \xi(x - A) + \eta(y - B) + \zeta(z - C) + \text{etc.} + M$$

quae eſt forma *ſecunda*.

Subſtituendo in forma ſecunda pro $x - A$, $y - B$, $z - C$ etc. expreſſiones VII. art. 21, obtinemus formam *tertiam*:

$$\Omega = [\alpha\alpha]\xi\xi + [\beta\beta]\eta\eta + [\gamma\gamma]\zeta\zeta + \text{etc.} + 2[\alpha\beta]\xi\eta + 2[\alpha\gamma]\xi\zeta + 2[\beta\gamma]\eta\zeta + \text{etc.} + M.$$

His adiungi poteſt forma *quarta*, ex forma tertia, atque formulis art. praec. ſponte demanans:

$$\Omega = (v - \lambda)^2 + (v' - \lambda')^2 + (v'' - \lambda'')^2 + \text{etc.} + M, \text{ ſiue}$$
$$\Omega = M + \Sigma(v - \lambda)^2$$

quae forma conditionem minimi directe ob oculos ſiſtit.

28.

Sint e, e', e'' etc. errores in obſeruationibus, quae dederunt $V = L$, $V' = L'$, $V'' = L''$ etc. commiſſi, i. e. ſint valores veri functionum V, V', V'' etc. reſp. $L - e$, $L' - e'$, $L'' - e''$ etc. adeoque valores veri ipſarum v, v', v'' etc. reſp. $-e\sqrt{p}$, $-e'\sqrt{p'}$,

$-e''\sqrt{p''}$ etc. Hinc valor verus ipsius x erit $= A - \alpha e\sqrt{p} - \alpha' e'\sqrt{p'} - \alpha'' e''\sqrt{p''}$ etc., siue error valoris ipsius x, in determinatione maxime idonea commissus, quem per Ex denotare conuenit,

$$= \alpha e\sqrt{p} + \alpha' e'\sqrt{p'} + \alpha'' e''\sqrt{p''} + \text{etc.}$$

Perinde error valoris ipsius y in determinatione maxime idonea commissus, quem per Ey denotabimus, erit

$$= \beta e\sqrt{p} + \beta' e'\sqrt{p'} + \beta'' e''\sqrt{p''} + \text{etc.}$$

Valor medius quadrati $(Ex)^2$ inuenitur $= mmp(\alpha\alpha + \alpha'\alpha' + \alpha''\alpha'' + \text{etc.}) = mmp[\alpha\alpha]$; valor medius quadrati $(Ey)^2$ perinde $= mmp[\beta\beta]$ etc., vt iam supra docuimus. Iam vero etiam valorem medium *producti* $Ex.Ey$ assignare licet, quippe qui invenitur

$$= mmp(\alpha\beta + \alpha'\beta' + \alpha''\beta'' + \text{etc.}) = mmp[\alpha\beta].$$

Concinne haec ita quoque exprimi possunt. Valores medii quadratorum $(Ex)^2$, $(Ey)^2$ etc. resp. aequales sunt productis ex $\frac{1}{2}mmp$ in quotientes differentialium partialium secundi ordinis

$$\frac{\mathrm{d}\,\mathrm{d}\Omega}{\mathrm{d}\xi^2},\ \frac{\mathrm{d}\,\mathrm{d}\Omega}{\mathrm{d}\eta^2}\ \text{etc.}$$

valorque medius producti talis, vt $Ex.Ey$, aequalis est producto ex $\frac{1}{2}mmp$ in quotientem differentialem $\frac{\mathrm{d}\,\mathrm{d}\Omega}{\mathrm{d}\xi.\mathrm{d}\eta}$, quatenus quidem Ω tamquam functio indeterminatarum ξ, η, ζ etc. consideratur.

29.

Designet t functionem datam linearem quantitatum x, y, z etc. puta sit

$$t = fx + gy + hz + \text{etc.} + k.$$

Valor ipsius t, e valoribus maxime plausibilibus ipsarum x, y, z etc. prodiens hinc erit $= fA + gB + hC + \text{etc.} + k$, quem per K denotabimus. Qui si tamquam valor verus ipsius t adoptatur, error committitur, qui erit

$$= fEx + gEy + hEz + \text{etc.}$$

atque per Et denotabitur. Manifesto valor medius huius erroris fit $= 0$, siue error a parte constante liber erit. At valor medius quadrati $(Et)^2$, siue valor medius aggregati

$$ff(Ex)^2 + 2fgEx.Ey + 2fhEx.Ez + \text{etc.}$$
$$+ gg(Ey)^2 + 2ghEy.Ez + \text{etc.}$$
$$+ hh(Ez)^2 + \text{etc. etc.}$$

per ea, quae in art. praec. exposuimus, aequalis fit producto ex mmp in aggregatum

$$ff[\alpha\alpha] + 2fg[\alpha\beta] + 2fh[\alpha\gamma] + \text{etc.}$$
$$+ gg[\beta\beta] + 2gh[\beta\gamma] + \text{etc.}$$
$$+ hh[\gamma\gamma] + \text{etc. etc.}$$

siue producto ex mmp in valorem functionis $\Omega - M$, qui prodit per substitutiones

$$\xi = f,\ \eta = g,\ \zeta = h \text{ etc.}$$

Denotando igitur hunc valorem determinatum functionis $\Omega - M$ per ω, error medius metuendus, dum determinationi $t = K$ adhaeremus, erit $= m\sqrt{p\omega}$, siue pondus huius determinationis $= \frac{1}{\omega}$.

Quum indefinite habeatur $\Omega - M = (x - A)\xi + (y - B)\eta + (z - C)\zeta + \text{etc.}$, patet, ω quoque aequalem esse valori determinato expressionis $(x - A)f + (y - B)g + (z - C)h + \text{etc.}$, siue valori determinato ipsius $t - K$, qui prodit, si indeterminatis x, y, z etc. tribuuntur valores ii, qui respondent valoribus ipsarum ξ, η, ζ etc. his f, g, h etc.

Denique obseruamus, si t indefinite in formam functionis ipsarum ξ, η, ζ etc. redigatur, ipsius partem constantem necessario fieri $= K$. Quodsi igitur indefinite fit

$$t = F\xi + G\eta + H\zeta + \text{etc.} + K$$

erit $\omega = fF + gG + hH + \text{etc.}$

30.

Functio Ω valorem suum *absolute minimum* M, vt supra vidimus, nanciscitur, faciendo $x = A$, $y = B$, $z = C$ etc., siue $\xi = 0$

$\eta = 0$, $\zeta = 0$ etc. Si vero alicui illarum quantitatum valor *alius* iam tributus eſt, e. g. $x = A + \Delta$, variantibus reliquis Ω aſſequi poteſt valorem relatiue minimum, qui manifeſto obtinetur adiumento aequationum

$$x = A + \Delta, \quad \frac{d\Omega}{dy} = 0, \quad \frac{d\Omega}{dx} = 0 \text{ etc.}$$

Fieri debet itaque $\eta = 0$, $\zeta = 0$ etc., adeoque, quoniam $x = A + [\alpha\alpha]\xi + [\alpha\beta]\eta + [\alpha\gamma]\zeta +$ etc., $\xi = \frac{\Delta}{[\alpha\alpha]}$. Simul habebitur

$$y = B + \frac{[\alpha\beta]\Delta}{[\alpha\alpha]}, \quad z = C + \frac{[\alpha\gamma]\Delta}{[\alpha\alpha]} \text{ etc.}$$

Valor relatiue minimus ipſius Ω autem fit $= [\alpha\alpha]\,\xi\xi + M = M + \frac{\Delta\Delta}{[\alpha\alpha]}$. Vice verſa hinc colligimus, ſi valor ipſius Ω limitem praeſcriptum $M + \mu\mu$ non ſuperare debet, valorem ipſius x neceſſario inter limites $A - \mu\sqrt{[\alpha\alpha]}$ et $A + \mu\sqrt{[\alpha\alpha]}$ contentum eſſe debere. Notari meretur, $\mu\sqrt{[\alpha\alpha]}$ aequalem fieri errori medio in valore maxime plauſibili ipſius x metuendo, ſi ſtatuatur $\mu = m\sqrt{p}$, i. e. ſi μ aequalis ſit errori medio obſeruationum talium, quibus pondus $= 1$ tribuitur.

Generalius inueſtigemus valorem minimum ipſius Ω, qui pro valore dato ipſius t locum habere poteſt, denotante t vt in art. praec. functionem linearem $fx + gy + hz +$ etc. $+ k$, et cuius valor maxime plauſibilis $= K$: valor praeſcriptus ipſius t denotetur per $K + \varkappa$. E theoria maximorum et minimorum conſtat, problematis ſolutionem petendam eſſe ex aequationibus

$$\frac{d\Omega}{dx} = \theta\frac{dt}{dx}$$
$$\frac{d\Omega}{dy} = \theta\frac{dt}{dy}$$
$$\frac{d\Omega}{dz} = \theta\frac{dt}{dz} \text{ etc.}$$

ſiue $\xi = \theta f$, $\eta = \theta g$, $\zeta = \theta h$ etc., deſignante θ multiplicatorem adhuc indeterminatum. Quare ſi, vt in art. praec., ſtatuimus, eſſe *indefinite*.

$$t = F\xi + G\eta + H\zeta + \text{etc.} + K$$

habebimus

$$K + \varkappa = \theta(fF + gG + hH + \text{etc.}) + K, \text{ ſiue}$$

$$\theta = \frac{\varkappa}{\omega}$$

accipiendo ω in eadem ſignificatione vt in art. praec. Et quum $\Omega - M$, indefinite, ſit functio homogenea ſecundi ordinis indeterminatarum ξ, η, ζ etc., ſponte patet, eius valorem pro $\xi = \theta f$, $\eta = \theta g$, $\zeta = \theta h$ etc. fieri $= \theta\theta\omega$, et proin valorem minimum, quem Ω pro $t = K + \varkappa$ obtinere poteſt, fieri $= M + \theta\theta\omega$ $= M + \frac{\varkappa\varkappa}{\omega}$. Vice verſa, ſi Ω debet valorem aliquem praeſcriptum $M + \mu\mu$ non ſuperare, valor ipſius t neceſſario inter limites $K - \mu\sqrt{\omega}$ et $K + \mu\sqrt{\omega}$ contentus eſſe debet, vbi $\mu\sqrt{\omega}$ aequalis ſit errori medio in determinatione maxime plauſibili ipſius t metiendo, ſi pro μ accipitur error medius obſeruationum, quibus pondus $= 1$ tribuitur.

31.

Quoties multitudo quantitatum x, y, z etc. paullo maior eſt, determinatio numerica valorum A, B, C etc. ex aequationibus $\xi = 0$, $\eta = 0$, $\zeta = 0$ etc. per eliminationem vulgarem ſatis moleſta euadit. Propterea in Theoria Motus Corporum Coeleſtium art. 182 algorithmum peculiarem addigitauimus, atque in *Disquiſitione de elementis ellipticis Palladis* (Comm. recent. Soc. Gotting. Vol. I.) copioſe explicauimus, per quem labor ille ad tantam quantam quidem res fert ſimplicitatem euehitur. Reducenda ſcilicet eſt functio Ω ſub formam talem:

$$\frac{u^0 u^0}{\mathfrak{A}^0} + \frac{u' u'}{\mathfrak{B}'} + \frac{u'' u''}{\mathfrak{C}''} + \frac{u''' u'''}{\mathfrak{D}'''} + \text{etc.} + M$$

vbi diuiſores $\mathfrak{A}^0$, $\mathfrak{B}'$, $\mathfrak{C}''$, $\mathfrak{D}'''$ etc. ſunt quantitates determinatae; u^0, u', u'', u''' etc. autem functiones lineares ipſarum x, y, z etc. quarum tamen ſecunda u' libera eſt ab x, tertia u'' libera ab x et y, quarta libera ab x, y et z, et ſic porro, ita vt vltima $u^{(\pi-1)}$ ſolam vltimam indeterminatarum x, y, z etc. implicet; denique coëfficientes, per quos x, y, z etc. reſp. multiplicatae ſunt in u^0, u', u'' etc., reſp. aequales ſunt ipſis $\mathfrak{A}^0$, $\mathfrak{B}'$, $\mathfrak{C}''$ etc. Quibus ita factis ſtatuendum eſt $u^0 = 0$, $u' = 0$, $u'' = 0$, $u''' = 0$ etc., vnde valores incognitarum x, y, z etc. inuerſo ordine commodiſſime elicientur. Haud opus videtur, algorithmum ipſum, per quem haec transformatio functionis Ω abſoluitur, hic denuo repetere.

Sed multo adhuc magis prolixum calculum requirit eliminatio indefinita, cuius adiumento illarum determinationum pondera inuenire oportet. Pondus quidem determinationis incognitae vltimae (quae ſola vltimam $u^{(\pi-1)}$ ingreditur) per ea, quae in Theoria Motus Corporum Coeleſtium demonſtrata ſunt, facile inuenitur aequale termino vltimo in ſerie diuiſorum $\mathfrak{A}^0$, $\mathfrak{B}'$, $\mathfrak{C}''$ etc.; quapropter plures calculatores, vt eliminationem illam moleſtam euitarent, deficientibus aliis ſubſidiis, ita ſibi conſuluerunt, vt algorithmum de quo diximus pluries, mutato quantitatum x, y, z etc., ordine, repeterent, ſingulis deinceps vltimum locum occupantibus. Gratum itaque geometris fore ſperamus, ſi modum nouum pondera determinationum calculandi, e penitiori argumenti perſcrutatione hauſtum hic exponamus, qui nihil amplius deſiderandum relinquere videtur.

32.

Statuamus itaque esse (I)

$$u^\circ = \mathfrak{A}^\circ x + \mathfrak{B}^\circ y + \mathfrak{C}^\circ z + \text{etc.} + \mathfrak{L}^\circ$$
$$u' = \mathfrak{B}' y + \mathfrak{C}' z + \text{etc.} + \mathfrak{L}'$$
$$u'' = \mathfrak{C}'' z + \text{etc.} + \mathfrak{L}''$$

etc.

Hinc erit indefinite

$$\tfrac{1}{2} d\Omega = \xi dx + \eta dy + \zeta dz + \text{etc.}$$
$$= \frac{u^\circ du^\circ}{\mathfrak{A}^\circ} + \frac{u' du'}{\mathfrak{B}'} + \frac{u'' du''}{\mathfrak{C}''} + \text{etc.}$$
$$= u^\circ \left(dx + \frac{\mathfrak{B}^\circ}{\mathfrak{A}^\circ} dy + \frac{\mathfrak{C}^\circ}{\mathfrak{A}^\circ} dz + \text{etc.}\right)$$
$$+ u'\left(dy + \frac{\mathfrak{C}'}{\mathfrak{B}'} dz + \text{etc.}\right) + u''(dz + \text{etc.}) + \text{etc.}$$

vnde colligimus (II)

$$\xi = u^\circ$$
$$\eta = \frac{\mathfrak{B}^\circ}{\mathfrak{A}^\circ} u^\circ + u'$$
$$\zeta = \frac{\mathfrak{C}^\circ}{\mathfrak{A}^\circ} u^\circ + \frac{\mathfrak{C}'}{\mathfrak{B}'} u' + \mu''$$

etc.

Supponamus, hinc deriuari formulas sequentes (III)

$$u^\circ = \xi$$
$$u' = A' \xi + \eta$$
$$u'' = A'' \xi + B'' \eta + \zeta$$

etc.

Iam e differentiali completo aequationis

$$\Omega = \xi(x - A) + \eta(y - B) + \zeta(z - C) + \text{etc.} + M$$

subtracta aequatione

$$\tfrac{1}{2} d\Omega = \xi dx + \eta dy + \zeta dz + \text{etc.}$$

sequitur

$$\tfrac{1}{2} d\Omega = (x - A) d\xi + (y - B) d\eta + (z - C) d\zeta + \text{etc.}$$

quae expreſſio identica eſſe debet cum hac ex III demanante:

$$\frac{u^\circ}{\mathfrak{A}^\circ}.\,d\xi + \frac{u'}{\mathfrak{B}'}(A'd\xi + d\eta) + \frac{u''}{\mathfrak{C}''}(A''d\xi + B''d\eta + d\zeta) + \text{etc.}$$

Hinc colligimus (IV)

$$x = \frac{u^\circ}{\mathfrak{A}^\circ} + A'.\frac{u'}{\mathfrak{B}'} + A''.\frac{u''}{\mathfrak{C}''} + \text{etc.} + A$$

$$y = \frac{u'}{\mathfrak{B}'} + B''.\frac{u''}{\mathfrak{C}''} + \text{etc.} + B$$

$$z = \frac{u''}{\mathfrak{C}''} + \text{etc.} + C$$

etc.

Subſtituendo in his expreſſionibus pro u°, u', u'' etc. valores earum ex III depromtos, eliminatio indefinita abſoluta erit. Et quidem ad pondera determinanda habebimus (V)

$$[\alpha\alpha] = \frac{1}{\mathfrak{A}^\circ} + \frac{A'A'}{\mathfrak{B}'} + \frac{A''A''}{\mathfrak{C}''} + \frac{A'''A'''}{\mathfrak{D}'''} + \text{etc.}$$

$$[\beta\beta] = \frac{1}{\mathfrak{B}'} + \frac{B''B''}{\mathfrak{C}''} + \frac{B'''B'''}{\mathfrak{D}'''} + \text{etc.}$$

$$[\gamma\gamma] = \frac{1}{\mathfrak{C}''} + \frac{C'''C'''}{\mathfrak{D}'''} + \text{etc.}$$

etc.

quarum formularum ſimplicitas nihil deſiderandum relinquit. Ceterum etiam pro coëfficientibus reliquis $[\alpha\beta]$, $[\alpha\gamma]$, $[\beta\gamma]$ etc. formulae aeque ſimplices prodeunt, quas tamen, quum illorum vſus ſit rarior, hic apponere ſuperſedemus.

33.

Propter rei grauitatem, et vt omnia ad calculum parata ſint, etiam formulas explicitas ad determinationem coëfficientium A', A'', A''' etc. B'', B''' etc. etc. hic adſcribere viſum eſt. Duplici modo hic calculus adornari poteſt, quum aequationes identicae prodire debeant, tum ſi valores ipſarum u°, u', u'' etc. ex III depromti in II ſubſtituuntur, tum ex ſubſtitutione valorum

η, ζ etc. ex II in III. Prior modus haec formularum
ubminiſtrat:

$$+ A' = 0$$

$$+ \frac{\mathfrak{C}'}{\mathfrak{A}'} \cdot A' + A'' = 0$$

$$+ \frac{\mathfrak{D}'}{\mathfrak{B}'} \cdot A' + \frac{\mathfrak{D}''}{\mathfrak{C}''} \cdot A'' + A''' = 0$$

inueniuntur A', A'', A''' etc.

$$+ B'' = 0$$

$$+ \frac{\mathfrak{D}''}{\mathfrak{C}''} B'' + B''' = 0$$

inueniuntur B'', B''' etc.

$$+ C''' = 0$$

inueniuntur C''' etc. Et ſic porro.

modus has formulas ſuggerit:

$$A' + \mathfrak{B}^\circ = 0$$

tur A'.

$$A'' + \mathfrak{B}^\circ B'' + \mathfrak{C}^\circ = 0$$

$$\mathfrak{B}' B'' + \mathfrak{C}' = 0$$

niuntur B'' et A''.

$$A''' + \mathfrak{B}^\circ B''' + \mathfrak{C}^\circ C''' + \mathfrak{D}^\circ = 0$$

$$\mathfrak{B}' B''' + \mathfrak{C}' C''' + \mathfrak{D}' = 0$$

$$\mathfrak{C}'' C''' + \mathfrak{D}'' = 0$$

miuntur C''', B''', A'''. Et ſic porro.

nodus aeque fere commodus eſt, ſi pondera determina-
nctarum x, y, z etc. deſiderantur; quoties vero e quan-
$\alpha\alpha]$, $[\beta\beta]$, $[\gamma\gamma]$ etc. vna tantum vel altera requiri-
ifeſto ſyſtema prius longe praeferendum erit.

quae expreſſio identica eſſe debet cum hac ex III demanante:

$$\frac{u^\circ}{\mathfrak{A}^\circ}\cdot d\xi + \frac{u'}{\mathfrak{B}'}(A' d\xi + d\eta) + \frac{u''}{\mathfrak{C}''}(A'' d\xi + B'' d\eta + d\zeta) + \text{etc.}$$

Hinc colligimus (IV)

$$x = \frac{u^\circ}{\mathfrak{A}^\circ} + A'.\frac{u'}{\mathfrak{B}'} + A''.\frac{u''}{\mathfrak{C}''} + \text{etc.} + A$$

$$y = \frac{u'}{\mathfrak{B}'} + B''.\frac{u''}{\mathfrak{C}''} + \text{etc.} + B$$

$$z = \frac{u''}{\mathfrak{C}''} + \text{etc.} + C$$

etc.

Subſtituendo in his expreſſionibus pro u°, u', u'' etc. valores earum ex III depromtos, eliminatio indefinita abſoluta erit. Et quidem ad pondera determinanda habebimus (V)

$$[\alpha\alpha] = \frac{1}{\mathfrak{A}^\circ} + \frac{A'A'}{\mathfrak{B}'} + \frac{A''A''}{\mathfrak{C}''} + \frac{A'''A'''}{\mathfrak{D}'''} + \text{etc.}$$

$$[\beta\beta] = \frac{1}{\mathfrak{B}'} + \frac{B''B''}{\mathfrak{C}''} + \frac{B'''B'''}{\mathfrak{D}'''} + \text{etc.}$$

$$[\gamma\gamma] = \frac{1}{\mathfrak{C}''} + \frac{C'''C'''}{\mathfrak{D}'''} + \text{etc.}$$

etc.

quarum formularum ſimplicitas nihil deſiderandum relinquit. Ceterum etiam pro coëfficientibus reliquis $[\alpha\beta]$, $[\alpha\gamma]$, $[\beta\gamma]$ etc. formulae aeque ſimplices prodeunt, quas tamen, quum illorum vſus ſit rarior, hic apponere ſuperſedemus.

33.

Propter rei grauitatem, et vt omnia ad calculum parata ſint, etiam formulas explicitas ad determinationem coëfficientium A', A'', A''' etc. B'', B''' etc. etc. hic adſcribere viſum eſt. Duplici modo hic calculus adornari poteſt, quum aequationes identicae prodire debeant, tum ſi valores ipſarum u°, u', u'' etc. ex III depromti in II ſubſtituuntur, tum ex ſubſtitutione valorum

ipſarum ξ, η, ζ etc. ex II in III. Prior modus haec formularum ſyſtemata ſubminiſtrat:

$$\frac{\mathfrak{B}^\circ}{\mathfrak{A}^\circ} + A' = 0$$

$$\frac{\mathfrak{C}^\circ}{\mathfrak{A}^\circ} + \frac{\mathfrak{C}'}{\mathfrak{A}'} . A' + A'' = 0$$

$$\frac{\mathfrak{D}^\circ}{\mathfrak{A}^\circ} + \frac{\mathfrak{D}'}{\mathfrak{B}'} . A' + \frac{\mathfrak{D}''}{\mathfrak{C}''} . A'' + A''' = 0$$

etc. vnde inueniuntur A', A'', A''' etc.

$$\frac{\mathfrak{C}'}{\mathfrak{B}'} + B'' = 0$$

$$\frac{\mathfrak{D}'}{\mathfrak{B}'} + \frac{\mathfrak{D}''}{\mathfrak{C}''} B'' + B''' = 0$$

etc. vnde inueniuntur B'', B''' etc.

$$\frac{\mathfrak{D}''}{\mathfrak{C}''} + C''' = 0$$

etc. vnde inueniuntur C''' etc. Et ſic porro.

Alter modus has formulas ſuggerit:

$$\mathfrak{A}^\circ A' + \mathfrak{B}^\circ = 0$$

vnde habetur A'.

$$\mathfrak{A}^\circ A'' + \mathfrak{B}^\circ B'' + \mathfrak{C}^\circ = 0$$
$$\mathfrak{B}' B'' + \mathfrak{C}' = 0$$

vnde inueniuntur B'' et A'',

$$\mathfrak{A}^\circ A''' + \mathfrak{B}^\circ B''' + \mathfrak{C}^\circ C''' + \mathfrak{D}^\circ = 0$$
$$\mathfrak{B}' B''' + \mathfrak{C}' C''' + \mathfrak{D}' = 0$$
$$\mathfrak{C}'' C''' + \mathfrak{D}'' = 0$$

vnde inueniuntur C''', B''', A'''. Et ſic porro.

Vterque modus aeque fere commodus eſt, ſi pondera determinationum cunctarum x, y, z etc. deſiderantur; quoties vero e quantitatibus $[\alpha\alpha]$, $[\beta\beta]$, $[\gamma\gamma]$ etc. vna tantum vel altera requiritur, manifeſto ſyſtema prius longe praeferendum erit.

Ceterum combinatio aequationum I cum IV ad easdem formulas perducit, insuperque calculum duplicem ad eruendos valores maxime plausibiles A, B, C etc. ipsos suppeditat, puta *primo*

$$A = -\frac{\xi^\circ}{\mathfrak{A}^\circ} - A'\frac{\xi'}{\mathfrak{B}'} - A''\frac{\xi''}{\mathfrak{C}''} - A'''\frac{\xi'''}{\mathfrak{D}'''} - \text{etc.}$$

$$B = \qquad -\frac{\xi'}{\mathfrak{B}'} - B''\frac{\xi''}{\mathfrak{C}''} - B'''\frac{\xi'''}{\mathfrak{D}'''} - \text{etc.}$$

$$C = \qquad\qquad -\frac{\xi''}{\mathfrak{C}''} - C'''\frac{\xi'''}{\mathfrak{D}'''} - \text{etc.}$$

etc.

Calculus alter identicus est cum vulgari, vbi statuitur $u^\circ = 0$, $u' = 0$, $u'' = 0$ etc.

34.

Quae in art. 32. exposuimus, sunt tantummodo casus speciales theorematis generalioris, quod ita se habet:

THEOREMA. *Designet t functionem linearem indeterminatarum x, y, z etc. hanc*

$$t = fx + gy + hz + \text{etc.} + k,$$

quae transmutata in functionem indeterminatarum u°, u', u'' etc fiat

$$t = k^\circ u^\circ + k'u' + k''u'' + \text{etc.} + K$$

Quibus ita se habentibus erit K valor maxime plausibilis ipsius t, atque pondus huius determinationis

$$= \frac{1}{\mathfrak{A}^\circ k^\circ k^\circ + \mathfrak{B}' k' k' + \mathfrak{C}'' k'' k'' + \text{etc.}}$$

Dem. Pars prior theorematis inde patet, quod valor maxime plausibilis ipsius t valoribus $u^\circ = 0$, $u' = 0$, $u'' = 0$ etc. respondere debet. Ad posteriorem demonstrandam obseruamus, quoniam $\frac{1}{2}\mathrm{d}\Omega = \xi\mathrm{d}x + \eta\mathrm{d}y + \zeta\mathrm{d}z + \text{etc.}$, atque $\mathrm{d}t = f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z + \text{etc.}$, esse, pro $\xi = f$, $\eta = g$, $\zeta = h$ etc., independenter a valoribus differentialium $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ etc.

$$\mathrm{d}\Omega = 2\,\mathrm{d}t$$

Hinc, vero sequitur, pro iisdem valoribus $\xi = f$, $\eta = g$, $\zeta = h$ etc. fieri

$$\frac{u^\circ}{\mathfrak{A}^\circ} \mathrm{d}u^\circ + \frac{u'}{\mathfrak{B}'} \mathrm{d}u' + \frac{u''}{\mathfrak{C}''} \mathrm{d}u'' + \text{etc.} = k^\circ \mathrm{d}u^\circ + k' \mathrm{d}u' + k'' \mathrm{d}u'' + \text{etc.}$$

Iam facile perspicitur, si $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ etc. sint ab inuicem independentes, etiam $\mathrm{d}u^\circ$, $\mathrm{d}u'$, $\mathrm{d}u''$ etc., ab inuicem independentes esse; vnde colligimus, pro $\xi = f$, $\eta = g$, $\zeta = h$ etc. esse

$$u^\circ = \mathfrak{A}^\circ k^\circ,\ u' = \mathfrak{B}' k',\ u'' = \mathfrak{C}'' k'' \text{ etc}$$

Quamobrem valor ipsius Ω, iisdem valoribus respondens erit

$$= \mathfrak{A}^\circ k^\circ k^\circ + \mathfrak{B}' k' k' + \mathfrak{C}'' k'' k'' + \text{etc.} + M.$$

vnde per art. 29. theorematis nostri veritas protinus demanat.

Ceterum si transformationem functionis t immediate, i. e. absque cognitione substitutionum IV. art. 32, perficere cupimus, praesto sunt formulae:

$$f = \mathfrak{A}^\circ k^\circ$$
$$g = \mathfrak{B}^\circ k^\circ + \mathfrak{B}' k'$$
$$h = \mathfrak{C}^\circ k^\circ + \mathfrak{C}' k' + \mathfrak{C}'' k''$$

etc., vnde coëfficientes k°, k', k'' etc. deinceps determinabuntur, tandemque habebitur

$$K = -\mathfrak{L}^\circ k^\circ - \mathfrak{L}' k' - \mathfrak{L}'' k'' - \text{etc.}$$

35.

Tractatione peculiari dignum est problema sequens, tum propter vtilitatem practicam, tum propter solutionis concinnitatem.

Inuenire mutationes valorum maxime plausibilium incognitarum ab accessione aequationis nouae productas, nec non pondera nouarum determinationum.

Retinebimus designationes in praecedentibus adhibitas, ita vt aequationes primitiuae, ad pondus $= 1$ reductae, sint hae

$v = 0$, $v' = 0$, $v'' = 0$ etc.; aggregatum indefinitum $vv + v'v' + v''v''$ etc. $= \Omega$; porro vt ξ, η, ζ etc. ſint quotientes differentiales partiales

$$\frac{d\Omega}{2dx}, \frac{d\Omega}{2dy}, \frac{d\Omega}{2dz} \text{ etc.}$$

denique vt ex eliminatione indefinita ſequatur

$$\left.\begin{aligned} x &= A + [\alpha\alpha]\xi + [\alpha\beta]\eta + [\alpha\gamma]\zeta + \text{etc.} \\ y &= B + [\alpha\beta]\xi + [\beta\beta]\eta + [\beta\gamma]\zeta + \text{etc.} \\ z &= C + [\alpha\gamma]\xi + [\beta\gamma]\eta + [\gamma\gamma]\zeta + \text{etc.} \end{aligned}\right\} \text{(I)}$$

Iam ſupponamus, accedere aequationem nouam $v^* = 0$ (proxime veram, et cuius pondus $= 1$), et inquiramus, quantas mutationes hinc nacturi ſint tum valores incognitarum maxime plauſibiles A, B, C etc., tum coëfficientes $[\alpha\alpha]$, $[\alpha\beta]$ etc.

Statuamus $\Omega + v^*v^* = \Omega^*$,

$$\frac{d\Omega^*}{2dx} = \xi^*, \frac{d\Omega^*}{2dy} = \eta^*, \frac{d\Omega^*}{2dz} = \zeta^* \text{ etc.}$$

ſupponamusque, hinc per eliminationem ſequi

$$x = A^* + [\alpha\alpha^*]\xi^* + [\alpha\beta^*]\eta^* + [\alpha\gamma^*]\zeta^* \text{ etc.}$$

Denique ſit

$$v^* = fx + gy + hz + \text{etc.} + k$$

prodeat inde, ſubſtitutis pro x, y, z etc. valoribus ex (I),

$$v^* = F\xi + G\eta + H\zeta + \text{etc.} + K$$

ſtatuaturque $Ff + Gg + Hh + \text{etc.} = \omega$.

Manifeſto K erit valor maxime plauſibilis functionis v^*, quatenus ex aequationibus primitiuis ſequitur, ſine reſpectu valoris 0 quem obſeruatio acceſſoria praebuit, atque $\frac{1}{\omega}$ pondus iſtius determinationis.

Iam habemus

$$\xi^* = \xi + fv^*, \ \eta^* = \eta + gv^*, \ \zeta^* = \zeta + hv^* \text{ etc.}$$

adeoque

$$F\xi^* + G\eta^* + H\zeta^* + \text{etc.} + K = v^*(1 + Ff + Gg + Hh + \text{etc.})$$

ſiue $v^* = \dfrac{F\xi^* + G\eta^* + H\zeta^* + \text{etc.} + K}{1+\omega}$

Perinde fit

$$x = A + [\alpha\alpha]\xi^* + [\alpha\beta]\eta^* + [\alpha\gamma]\zeta^* + \text{etc.} - v^*(f[\alpha\alpha] + g[\alpha\beta] + h[\alpha\gamma] + \text{etc.})$$

$$= A + [\alpha\alpha]\xi^* + [\alpha\beta]\eta^* + [\alpha\gamma]\zeta^* + \text{etc.} - Fv^*$$

$$= A + [\alpha\alpha]\xi^* + [\alpha\beta]\eta^* + [\alpha\gamma]\zeta^* + \text{etc.} - \frac{F}{1+\omega}(F\xi^* + G\eta^* + H\zeta^* + \text{etc.} + K)$$

Hinc itaque colligimus

$A^* = A - \dfrac{FK}{1+\omega}$, qui erit valor maxime plauſibilis ipſius x ex *omnibus* obſeruationibus;

$$[\alpha\alpha^*] = [\alpha\alpha] - \frac{FF}{1+\omega}$$

adeoque pondus iſtius determinationis

$$= \frac{1}{[\alpha\alpha] - \frac{FF}{1+\omega}}$$

Prorſus eodem modo inuenitur valor maxime plauſibilis ipſius y, *omnibus* obſeruationibus ſuperſtructus

$$B^* = B - \frac{GK}{1+\omega}$$

atque pondus huius determinationis

$$= \frac{1}{[\beta\beta] - \frac{GG}{1+\omega}}$$

et ſic porro. *Q. E. I.*

Liceat huic ſolutioni quasdam annotationes adiicere.

I. Subſtitutis his nouis valoribus A^*, B^*, C^* etc., functio v^* obtinet valorem maxime plauſibilem

$$K - \frac{K}{1+\omega}(Ff + Gg + Hh + \text{etc.}) = \frac{K}{1+\omega}.$$ Et quum indefinite ſit

$$v^* = \frac{F}{1+\omega}.\xi^* + \frac{G}{1+\omega}.\eta^* + \frac{H}{1+\omega}.\zeta^* + \text{etc.} + \frac{K}{1+\omega}$$

pondus iſtius determinationis per principia art. 29. eruitur

$$= \frac{1+\omega}{Ff + Gg + Hh + \text{etc.}} = \frac{1}{\omega} + 1.$$

Eadem immediate reſultant ex applicatione regulae in fine art 21. traditae; ſcilicet complexus aequationum primitiuarum praebuerat determinationem $v^* = K$ cum pondere $= \frac{1}{\omega}$, dein obſeruatio nova dedit determinationem aliam, ab illa independentem, $v^* = 0$, cum pondere $= 1$, quibus combinatis prodit determinatio $v^* = \frac{K}{1+\omega}$ cum pondere $= \frac{1}{\omega} + 1$.

II. Hinc porro ſequitur, quum pro $x = A^*$, $y = B^*$, $z = C^*$ etc. eſſe debeat $\xi^* = 0$, $\eta^* = 0$, $\zeta^* = 0$ etc., pro iisdem valoribus fieri

$$\xi = -\frac{fK}{1+\omega}, \ \eta = -\frac{gK}{1+\omega}, \ \zeta = -\frac{hK}{1+\omega} \text{ etc.}$$

nec non, quoniam indefinite $\Omega = \xi(x - A) + \eta(y - B) + g(z - C) + \text{etc.} + M$,

$$\Omega = \frac{KK}{(1+\omega)^2}(Ff + Gg + Hh + \text{etc.}) + M = M + \frac{\omega KK}{(1+\omega)^2};$$

denique, quoniam indefinite $\Omega^* = \Omega + v^* v^*$,

$$\Omega^* = M + \frac{\omega KK}{(1+\omega)^2} + \frac{KK}{(1+\omega)^2} = M + \frac{KK}{1+\omega}$$

III. Comparando haec cum iis quae in art. 30. docuimus, animaduertimus, functionem Ω hic valorem minimum obtinere, quem pro valore determinato functionis $v^* = \frac{K}{1+\omega}$ accipere poteſt.

36.

Problematis alius, praecedenti affinis, puta

Inuestigare mutationes valorum maxime plausibilium incognitarum, a mutato pondere vnius ex obseruationibus primitiuis oriundas, nec non pondera nouarum determinationum

solutionem tantummodo hic adscribemus, demonstrationem, quae ad instar art. praec. facile absoluitur, breuitatis caussa supprimentes.

Supponamus, peracto demum calculo animaduerti, alicui obseruationum pondus seu nimis paruum, seu nimis magnum tributum esse, e. g. obseruationi primae, quae dedit $V=L$, loco ponderis p in calculo adhibiti rectius tribui pondus $=p^*$. Tunc haud opus erit calculum integrum repetere, sed commodius correctiones per formulas sequentes computare licebit.

Valores incognitarum maxime plausibiles correcti erunt hi:

$$x=A-\frac{(p^*+p)\alpha\lambda}{p+(p^*-p)(a\alpha+b\beta+c\gamma+\text{etc.})}$$

$$y=B-\frac{(p^*-p)\beta\lambda}{p+(p^*-p)(a\alpha+b\beta+c\gamma+\text{etc.})}$$

$$z=C-\frac{(p^*-p)\gamma\lambda}{p+(p^*-p)(a\alpha+b\beta+c\gamma+\text{etc.})}$$

etc. ponderaque harum determinationum inuenientur, diuidendo vnitatem resp. per

$$[\alpha\alpha]-\frac{(p^*-p)\alpha\alpha}{p+(p^*-p)(a\alpha+b\beta+c\gamma+\text{etc.})}$$

$$[\beta\beta]-\frac{(p^*-p)\beta\beta}{p+(p^*-p)(a\alpha+b\beta+c\gamma+\text{etc.})}$$

$$[\gamma\gamma]-\frac{(p^*-p)\gamma\gamma}{p+(p^*-p)(a\alpha+b\beta+c\gamma+\text{etc.})}\ \text{etc.}$$

Haec solutio simul complectitur casum, vbi peracto calculo percipitur, vnam ex obseruationibus omnino reiici debuisse, quum hoc idem sit ac si facias $p^*=0$; et perinde valor $p^*=\infty$ refer-

tur ad casum eum, vbi aequatio $V = L$, quae in calculo tamquam approximata tractata erat, reuera praecisione absoluta gaudet.

Ceterum quoties vel aequationibus, quibus calculus superstructus erat, *plures* nouae accedunt, vel *pluribus* ex illis pondera erronea tributa esse percipitur, computus correctionum nimis complicatus euaderet; quocirca in tali casu calculum ab integro reficere praestabit.

37.

In art. 15, 16. methodum explicauimus, obseruationum praecisionem proxime determinandi *). Sed haec methodus supponit, errores, qui reuera occurrerint, satis multos exacte cognitos esse, quae conditio, stricte loquendo, rarissime, ne dicam numquam, locum habebit. Quodsi quidem quantitates, quarum valores approximati per obseruationes innotuerunt, secundum legem cognitam, ab vna pluribusue quantitatibus incognitis pendent, harum valores maxime plausibiles per methodum quadratorum minimorum eruere licebit, ac dein valores quantitatum, quae obseruationum obiecta fuerant, illinc computati perparum a valoribus veris discrepare censebuntur, ita vt ipsorum differentias a valoribus obseruatis eo maiori iure tamquam obseruationum errores veros adoptare liceat, quo maior fuerit harum multitudo. Hanc praxin sequuti sunt omnes calculatores, qui observationum praecisionem in casibus concretis a posteriori aestimare susceperunt: sed manifesto illa theoretice erronea est, et quamquam in casibus multis ad vsus practicos sufficere possit, tamen

*) Disquisitio de eodem argumento, quam in commentatione anteriori (*Zeitschrift für Astronomie und verwandte Wissenschaften* Vol. I, p. 185.) tradideramus, eidem hypothesi circa indolem functionis probabilitatem errorum exprimentis innixa erat, cui in Theoria motus corporum coelestium methodum quadratorum minimorum superstruxeramus (vid. art. 9, III.).

in aliis enormiter peccare poteſt. Summopere itaque hoc argumentum dignum eſt, quod accuratius enodetur.

Retinebimus in hac disquiſitione deſignationes inde ab art. 19. adhibitas. Praxis ea de qua diximus, quantitates A, B, C etc. tamquam valores veros ipſarum x, y, z conſiderat, et pro in ipſas λ, λ', λ'' etc. tamquam valores veros functionum v, v', v'' etc. Si omnes obſeruationes aequali praeciſione gaudent, ipſarumque pondus $p = p' = p''$ etc. pro vnitate acceptum eſt, eaedem quantitates, ſignis mutatis, in illa ſuppoſitione obſeruationum errores exhibent, vnde praecepta art. 15, praebent obſeruationum errorem medium m

$$= \sqrt{\frac{\lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' + \text{etc.}}{\pi}} = \sqrt{\frac{M}{\pi}}$$

Si obſeruationum praeciſio non eſt eadem, quantitates $-\lambda$, $-\lambda'$, $-\lambda''$ etc. exhiberent obſeruationum errores per radices quadratas e ponderibus multiplicatos, praeceptaque art. 16. ad eandem formulam $\sqrt{\frac{M}{\pi}}$ perducerent, iam errorem medium talium obſervationum, quibus pondus $= 1$ tribuitur, denotantem. Sed manifeſto calculus exactus requireret, vt loco quantitatum λ, λ', λ'' etc. valores functionum v, v', v'' etc. e valoribus veris ipſarum x, y, z etc. prodeuntes adhiberentur, i. e. loco ipſius M, valor functionis Ω valoribus veris ipſarum x, y, z etc. reſpondens. Qui quamquam aſſignari nequeat, tamen certi ſumus, eum eſſe maiorem quam M (quippe qui eſt minimus poſſibilis), excipiendo caſum infinite parum probabilem, vbi incognitarum valores maxime plauſibiles exacte cum veris quadrant. In genere itaque affirmare poſſumus, praxin vulgarem errorem medium iuſto minorem producere, ſiue obſeruationibus praeciſionem nimis magnam tribuere. Videamus iam, quid doceat theoria rigoroſa.

38.

Ante omnia inueſtigare oportet, quonam modo M ab obſervationum erroribus veris pendeat. Denotemus hos, vt in art. 28, per e, e', e'' etc., ſtatuamusque ad maiorem ſimplicitatem

$$e\sqrt{p}=\varepsilon,\ e'\sqrt{p'}=\varepsilon',\ e''\sqrt{p''}=\varepsilon'' \text{ etc., nec non}$$
$$m\sqrt{p}=m'\sqrt{p'}=m''\sqrt{p''} \text{ etc.} =\mu$$

Porro ſint valores veri ipſarum x, y, z etc. reſp. $A-x^\circ$, $B-y^\circ$, $C-z^\circ$ etc., quibus reſpondeant valores ipſarum ξ, η, ζ etc. hi $-\xi^\circ$, $-\eta^\circ$, $-\zeta^\circ$ etc. Manifeſto iisdem reſpondebunt valores ipſarum v, v', v'' etc. hi $-\varepsilon$, $-\varepsilon'$, $-\varepsilon''$ etc. ita vt habeatur

$$\xi^\circ=a\varepsilon+a'\varepsilon'+a''\varepsilon''+\text{ etc.}$$
$$\eta^\circ=b\varepsilon+b'\varepsilon'+b''\varepsilon''+\text{ etc.}$$
$$\zeta^\circ=c\varepsilon+c'\varepsilon'+c''\varepsilon''+\text{ etc.}$$

etc. nec non

$$x^\circ=\alpha\varepsilon+\alpha'\varepsilon'+\alpha''\varepsilon''+\text{ etc.}$$
$$y^\circ=\beta\varepsilon+\beta'\varepsilon'+\beta''\varepsilon''+\text{ etc.}$$
$$z^\circ=\gamma\varepsilon+\gamma'\varepsilon'+\gamma''\varepsilon''+\text{ etc.}$$

Denique ſtatuemus

$$\Omega^\circ=\varepsilon\varepsilon+\varepsilon'\varepsilon'+\varepsilon''\varepsilon''+\text{ etc.}$$

ita vt ſit Ω° aequalis valori functionis Ω valoribus veris ipſarum x, y, z etc. reſpondenti. Hinc quum habeatur indefinite $\Omega=M+(x-A)\xi+(y-B)\eta+(z-C)\zeta+$ etc., erit etiam

$$M=\Omega^\circ-x^\circ\xi^\circ-y^\circ\eta^\circ-z^\circ\zeta^\circ-\text{ etc.}$$

Hinc manifeſtum eſt, M, euolutione facta eſſe functionem hogeneam ſecundi ordinis errorum e, e', e'' etc., quae, pro diuerſis errorum valoribus maior minorue euadere poterit. Sed quatenus errorum magnitudo nobis incognita manet, functionem hanc indefinite conſiderare, imprimisque ſecundum principia calculi probabilitatis eius valorem medium aſſignare conueniet. Quem inueniemus, ſi loco quadratorum ee, $e'e'$, $e''e''$ etc. reſp. ſcribimus mm, $m'm'$, $m''m''$ etc., producta vero ee', ee'', $e'e''$ etc.

omnino omittimus, vel quod idem eſt, ſi loco cuiusuis quadrati $\varepsilon\varepsilon$, $\varepsilon'\varepsilon'$, $\varepsilon''\varepsilon''$ etc. ſcribimus $\mu\mu$, productis $\varepsilon\varepsilon'$, $\varepsilon\varepsilon''$ $\varepsilon'\varepsilon''$ etc. prorſus neglectis. Hoc modo e termino Ω° manifeſto prouenit $\pi\mu\mu$; terminus $-x^\circ\xi^\circ$ producet

$$-(a\alpha + a'\alpha' + a''\alpha'' + \text{etc.})\,\mu\mu = -\mu\mu$$

et ſimiliter ſingulae partes reliquae praebebunt $-\mu\mu$, ita vt valor medius totalis fiat $=(\pi-\varrho)\mu\mu$, denotante π multitudinem obſeruationum, ϱ multitudinem incognitarum. Valor verus quidem ipſius M, prout fors errores obtulit, maior minorue medio fieri poteſt, ſed diſcrepantia eo minoris momenti erit, quo maior fuerit obſeruationum multitudo, ita vt pro valore approximato ipſius μ accipere liceat

$$\sqrt{\frac{M}{\pi-\varrho}}$$

Valori itaque ipſius μ, ex praxi erronea, de qua in art. praec. loquuti ſumus, prodiens, augeri debet in ratione quantitatis $\sqrt{(\pi-\varrho)}$ ad $\sqrt{\pi}$.

39.

Quo clarius eluceat, quanto iure valorem fortuitum ipſius M medio aequiparare liceat, adhuc inueſtigare oportet errorem medium metuendum, dum ſtatuimus $\frac{M}{\pi-\varrho} = mm$. Iſte error medius aequalis eſt radici quadratae e valore medio quantitatis

$$\left(\frac{\Omega^\circ - x^\circ\xi^\circ - y^\circ\eta^\circ - z^\circ\zeta^\circ - \text{etc.} - (\pi-\varrho)mm}{\pi-\varrho}\right)^2$$

quam ita exhibebimus

$$\left(\frac{\Omega^\circ - x^\circ\xi^\circ - y^\circ\eta^\circ - z^\circ\zeta^\circ - \text{etc.}}{\pi-\varrho}\right)^2$$

$$-\frac{2\mu\mu}{\pi-\varrho}\left(\Omega^\circ - x^\circ\xi^\circ - y^\circ\eta^\circ - z^\circ\zeta^\circ - \text{etc.} - (\pi-\varrho)\mu\mu\right) - \mu^4$$

et quum manifeſto valor medius termini ſecundi fiat $=0$, res in eo vertitur, vt indagemus valorem medium functionis

$$\Psi = (\Omega^\circ - x^\circ \xi^\circ - y^\circ \eta^\circ - z^\circ \zeta^\circ - \text{etc.})^2$$

quo inuento et per N designato, error medius quaesitus erit

$$= \sqrt{\left(\frac{N}{(\pi - \varrho)^2} - \mu^4\right)}$$

Expressio Ψ euoluta manifesto est functio homogenea siue errorum e, e', e'' etc., siue quantitatum $\varepsilon, \varepsilon', \varepsilon''$ etc., eiusque valor medius inuenietur, si

1° pro biquadratis e^4, e'^4, e''^4 etc. substituuntur eorum valores medii

2° pro singulis productis e binis quadratis vt $ee\,e'e'$, $ee\,e''e''$, $e'e'\,e''e''$ etc. producta ex ipsorum valoribus mediis, puta $mm\,m'm'$, $mm\,m''m''$, $m'm'\,m''m''$ etc.

3° partes vero reliquae, quae implicabunt vel factorem talem $e^3 e'$, vel talem $ee\,e'e''$, omnino omittuntur. Valores medios biquadratorum e^4, e'^4, e''^4 etc. ipsis biquadratis m^4, m'^4, m''^4 etc. proportionales supponemus (vid. art. 16), ita vt illi sint ad haec vt ν^4 ad μ^4, adeoque ν^4 denotet valorem medium biquadratorum obseruationum talium quarum pondus $= 1$. Hinc praecepta praecedentia ita quoque exprimi poterunt: Loco singulorum biquadratorum $\varepsilon^4, \varepsilon'^4, \varepsilon''^4$ etc. scribendum erit ν^4, loco singulorum productorum e binis quadratis vt $\varepsilon\varepsilon\,\varepsilon'\varepsilon'$, $\varepsilon\varepsilon\,\varepsilon''\varepsilon''$, $\varepsilon'\varepsilon'\,\varepsilon''\varepsilon''$ etc., scribendum erit μ^4, omnesque reliqui termini, qui implicabunt factores tales vt $\varepsilon^3\varepsilon'$, vel $\varepsilon\varepsilon\,\varepsilon'\varepsilon''$, vel $\varepsilon\,\varepsilon'\varepsilon''\varepsilon'''$ erunt supprimendi.

His probe intellectis facile patebit

I. Valorem medium quadrati $\Omega^\circ\Omega^\circ$ esse $\pi\nu^4 + (\pi\pi - \pi)\mu^4$

II. Valor medius producti $\varepsilon\varepsilon x^\circ \xi^\circ$ fit $= \alpha\alpha\nu^4 + (\alpha'\alpha' + \alpha''\alpha'' + \text{etc.})\mu^4$, siue quoniam $\alpha\alpha + \alpha'\alpha' + \alpha''\alpha'' + \text{etc.} = 1$.

$$= \alpha\alpha(\nu^4 - \mu^4) + \mu^4$$

Et quum perinde valor medius producti $\varepsilon'\varepsilon' x^\circ \xi^\circ$ fiat =

$a'\alpha'(\nu^4-\mu^4)+\mu^4$, valor medius producti $\varepsilon''\varepsilon''x^\circ\xi^\circ$ autem $=a''\alpha''(\nu^4-\mu^4)+\mu^4$ et ſic porro, patet, valorem medium producti $(\varepsilon\varepsilon+\varepsilon'\varepsilon'+\varepsilon''\varepsilon''+\text{etc.})\,x^\circ\xi^\circ$ ſiue $\Omega^\circ x^\circ\xi^\circ$ eſſe

$$=\nu^4-\mu^4+\pi\mu^4$$

Eundem valorem medium habebunt producta $\Omega^\circ y^\circ\eta^\circ$, $\Omega^\circ z^\circ\zeta^\circ$ etc. Quapropter valor medius producti $\Omega^\circ(x^\circ\xi^\circ+y^\circ\eta^\circ+z^\circ\zeta^\circ+\text{etc.})$ fit

$$=\varrho\nu^4+\varrho(\pi-1)\mu^4$$

III. Ne euolutiones reliquae nimis prolixae euadant, idonea denotatio introducenda erit. Vtemur itaque characteriſtica Σ ſenſu aliquantum latiori quam ſupra paſſim factum eſt, ita vt denotet aggregatum termini, cui praefixa eſt, cum omnibus ſimilibus ſed non identicis inde per omnes obſeruationum permutationes oriundis. Hoc pacto e. g. habemus $x^\circ=\Sigma\alpha\varepsilon$, $x^\circ x^\circ=\Sigma\alpha\alpha\varepsilon\varepsilon+2\Sigma\alpha\alpha'\varepsilon\varepsilon'$. Colligendo itaque valorem medium producti $x^\circ x^\circ\xi^\circ\xi^\circ$ per partes, habemus primo valorem medium producti $\alpha\alpha\varepsilon\varepsilon\xi^\circ\xi^\circ$

$$=aa\alpha\alpha\nu^4+\alpha\alpha(a'a'+a''a''+\text{etc.})\mu^4$$
$$=aa\alpha\alpha(\nu^4-\mu^4)+\alpha\alpha\mu^4\Sigma aa$$

Perinde valor medius producti $\alpha'\alpha'\varepsilon'\varepsilon'\xi^\circ\xi^\circ$ fit $=a'a'\alpha'\alpha'(\nu^4-\mu^4)+\alpha'\alpha'\mu^4\Sigma aa$ et ſic porro, adeoque valor medius producti $\xi^\circ\xi^\circ\Sigma\alpha\alpha\varepsilon\varepsilon$

$$=(\nu^4-\mu^4)\Sigma aa\alpha\alpha+\mu^4\Sigma aa.\Sigma\alpha\alpha$$

Porro valor medius producti $\alpha\alpha'\varepsilon\varepsilon'\xi^\circ\xi^\circ$ ſit $=2\alpha\alpha'aa'\mu^4$, valor medius producti $\alpha\alpha''\varepsilon\varepsilon''\xi^\circ\xi^\circ$ perinde $=2\alpha\alpha''aa''\mu^4$ etc., vnde facile concluditur, valorem medium producti $\xi^\circ\xi^\circ\Sigma\alpha\alpha'\varepsilon\varepsilon'$ fieri

$$=2\mu^4\Sigma\alpha a\alpha'a'=\mu^4((\Sigma a\alpha)^2-\Sigma aa\alpha\alpha)=\mu^4(1-\Sigma aa\alpha\alpha)$$

His collectis habemus valorem medium producti $x^\circ x^\circ\xi^\circ\xi^\circ$

$$=(\nu^4-3\mu^4)\Sigma aa\alpha\alpha+2\mu^4+\mu^4\Sigma aa.\Sigma\alpha\alpha.$$

IV Haud abſimili modo inuenitur valor medius producti $x^0 y^0 \xi^0 \eta^0$

$$= \nu^4 \Sigma ab\alpha\beta + \mu^4 \Sigma a\alpha b'\beta' + \mu^4 \Sigma ab\alpha'\beta' + \mu^4 \Sigma a\beta b'\alpha'$$

Sed habetur

$$\Sigma a\alpha b'\beta' = \Sigma a\alpha . \Sigma b\beta - \Sigma a\alpha b\beta$$
$$\Sigma ab\alpha'\beta' = \Sigma ab . \Sigma\alpha\beta - \Sigma ab\alpha\beta$$
$$\Sigma a\beta b'\alpha' = \Sigma a\beta . \Sigma b\alpha - \Sigma a\beta b\alpha$$

vnde valor ille medius fit, propter $\Sigma a\alpha = 1$, $\Sigma b\beta = 1$, $\Sigma a\beta = 0$, $\Sigma b\alpha = 0$,

$$= (\nu^4 - 3\mu^4)\Sigma ab\alpha\beta + \mu^4(1 + \Sigma ab . \Sigma\alpha\beta)$$

V. Quum prorſus eodem modo valor medius producti $x^0 z^0 \xi^0 \zeta^0$ fiat

$$= (\nu^4 - 3\mu^4)\Sigma ac\alpha\gamma + \mu^4(1 + \Sigma ac . \Sigma\alpha\gamma)$$

et ſic porro, additio valorem medium producti $x^0 \xi^0 (x^0 \xi^0 + y^0 \eta^0 + z^0 \zeta^0 + \text{etc.})$ ſuppeditat

$$= (\nu^4 - 3\mu^4)\Sigma(a\alpha(a\alpha + b\beta + c\gamma + \text{etc.})) + (\varrho + 1)\mu^4$$
$$+ \mu^4(\Sigma aa . \Sigma\alpha\alpha + \Sigma ab . \Sigma\alpha\beta + \Sigma ac . \Sigma\alpha\gamma + \text{etc.})$$
$$= (\nu^4 - 3\mu^4)\Sigma(a\alpha(a\alpha + b\beta + c\gamma + \text{etc.})) + (\varrho + 2)\mu^4$$

VI. Prorſus eodem modo valor medius producti $y^0 \eta^0 (x^0 \xi^0 + z^0 \zeta^0 + \text{etc.})$ eruitur

$$= (\nu^4 - 3\mu^4)\Sigma(b\beta(a\alpha + b\beta + c\gamma + \text{etc.})) + (\varrho + 2)\mu^4$$

dein valor medius producti $z^0 \zeta^0 (x^0 \xi^0 + y^0 \eta^0 + z^0 \zeta^0 + \text{etc.})$

$$= (\nu^4 - 3\mu^4)\Sigma(c\gamma(a\alpha + b\beta + c\gamma + \text{etc.})) + (\varrho + 2)\mu^4$$

et ſic porro. Hinc per additionem prodit valor medius quadrati $(x^0 \xi^0 + y^0 \eta^0 + z^0 \zeta^0 + \text{etc.})^2$

$$= (\nu^4 - 3\mu^4)\Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2) + (\varrho\varrho + 2\varrho)\mu^4$$

VII. Omnibus tandem rite collectis eruitur

$$N = (\pi - 2\varrho)\nu^4 + (\pi\pi - \pi - 2\pi\varrho + 4\varrho + \varrho\varrho)\mu^4 +$$
$$(\nu^4 - 3\mu^4)\Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2)$$

$$=(\pi-\varrho)(\nu^4-\mu^4)+(\pi-\varrho)^2\mu^4-(\nu^4-3\mu^4)(\varrho-\Sigma((a\alpha+b\beta+c\gamma+\text{etc.})^2))$$

Error itaque medius in determinatione ipsius $\mu\mu$ per formulam

$$\mu\mu=\frac{M}{\pi-\varrho}$$

metuendus erit

$$=\sqrt{\left\{\frac{\nu^4-\mu^4}{\pi-\varrho}-\frac{\nu^4-3\mu^4}{(\pi-\varrho)^2}\cdot(\varrho-\Sigma((a\alpha+b\beta+c\gamma+\text{etc.})^2))\right\}}$$

40.

Quantitas $\Sigma((a\alpha+b\beta+c\gamma+\text{etc.})^2)$, quae in expressionem modo inuentam ingreditur, generaliter quidem ad formam simpliciorem reduci nequit: nihilominus duo limites assignari possunt, inter quos ipsius valor necessario iacere debet. *Primo* scilicet e relationibus supra euolutis facile demonstratur esse

$$(a\alpha+b\beta+c\gamma+\text{etc.})^2+(a\alpha'+b\beta'+c\gamma'+\text{etc.})^2+(a\alpha''+b\beta''+c\gamma''+\text{etc.})^2+\text{etc.}=a\alpha+b\beta+c\gamma+\text{etc.}$$

vnde concludimus, $a\alpha+b\beta+c\gamma+\text{etc.}$ esse quantitatem positiuam vnitate minorem (saltem non maiorem). Idem valet de quantitate $a'\alpha'+b'\beta'+c'\gamma'+\text{etc.}$, quippe cui aggregatum

$$(a'\alpha+b'\beta+c'\gamma+\text{etc.})^2+(a'\alpha'+b'\beta'+c'\gamma'+\text{etc.})^2+(a'\alpha''+b'\beta''+c'\gamma''\text{ etc.})^2+\text{etc.}$$

aequale inuenitur; ac perinde $a''\alpha''+b''\beta''+c''\gamma''+\text{etc.}$ vnitate minor erit, et sic porro. Hinc $\Sigma((a\alpha+b\beta+c\gamma+\text{etc.})^2)$ necessario est minor quam π. *Secundo* habetur $\Sigma(a\alpha+b\beta+c\gamma+\text{etc.})=\varrho$, quoniam fit $\Sigma a\alpha=1$, $\Sigma b\beta=1$, $\Sigma c\gamma=1$ etc; vnde facile deducitur, summam quadratorum $\Sigma((a\alpha+b\beta+c\gamma+\text{etc.})^2)$ esse maiorem quam $\frac{\varrho\varrho}{\pi}$, vel saltem non minorem. Hinc terminus

$$\frac{\nu^4 - 3\mu^4}{(\pi - \varrho)^2} \cdot (\varrho - \Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2))$$

neceſſario iacet inter limites $-\frac{\nu^4 - 3\mu^4}{\pi - \varrho}$ et $\frac{\nu^4 - 3\mu^4}{\pi - \varrho} \cdot \frac{\varrho}{\pi}$ vel, ſi latiores praeferimus, inter hos $-\frac{\nu^4 - 3\mu^4}{\pi - \varrho}$ et $+\frac{\nu^4 - 3\mu^4}{\pi - \varrho}$, et proin erroris medii in valore ipſius $\mu\mu = \frac{M}{\pi - \varrho}$ metuendi quadratum inter limites $\frac{2\nu^4 - 4\mu^4}{\pi - \varrho}$ et $\frac{2\mu^4}{\pi - \varrho}$, ita vt praeciſionem quantamuis aſſequi liceat, ſi modo obſeruationum multitudo fuerit ſatis magna.

Valde memorabile eſt, in hypotheſi ea (art 9, III.), cui theoria quadratorum minimorum olim ſuperſtructa fuerat, illum terminum omnino excidere, et ſicuti, ad eruendum valorem approximatum erroris medii obſeruationum μ, in omnibus caſibus aggregatum $\lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' + \text{etc.} = M$ ita tractare oportet, ac ſi eſſet aggregatum $\pi - \varrho$ errorum fortuitorum, ita in illa hypotheſi etiam praeciſionem ipſam huius determinationis aequalem fieri ei, quam determinationi ex $\pi - \varrho$ erroribus veris tribuendam eſſe in art. 15. inuenimus.